

spine.1 Defining the Stages as the V_α s

sth:spine:valpha:
sec

In ??, we defined well-orderings and the (von Neumann) ordinals. In this chapter, we will use these to characterise the hierarchy of sets *itself*. To do this, recall that in ??, we defined the idea of successor and limit ordinals. We use these ideas in following definition:

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defValphas

Definition spine.1.

$$\begin{aligned} V_\emptyset &:= \emptyset \\ V_{\alpha+} &:= \wp(V_\alpha) && \text{for any ordinal } \alpha \\ V_\alpha &:= \bigcup_{\gamma < \alpha} V_\gamma && \text{when } \alpha \text{ is a limit ordinal} \end{aligned}$$

This will be a definition by *transfinite recursion* on the ordinals. In this regard, we should compare this with recursive definitions of functions on the natural numbers.¹ As when dealing with natural numbers, one defines a base case and successor cases; but when dealing with ordinals, we also need to describe the behaviour of *limit* cases.

This definition of the V_α s will be an important milestone. We have informally motivated our hierarchy of sets as forming sets by *stages*. The V_α s are, in effect, just those stages. Importantly, though, this is an *internal* characterisation of the stages. Rather than suggesting a possible *model* of the theory, we will have defined the stages *within* our set theory.

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Bibliography

¹Cf. the definitions of addition, multiplication, and exponentiation in ??.