spine.1 Foundation

sth:spine:foundation:sec We are only *almost* done—and not *quite* finished—because nothing in $\mathbf{ZF}^-$ guarantees that *every* set is in some V_α, i.e., that every set is formed at some stage.

Now, there is a fairly straightforward (mathematical) sense in which we don't *care* whether there are sets outside the hierarchy. (If there are any there, we can simply ignore them.) But we have motivated our *concept* of set with the thought that every set is formed at some stage (see *Stages-are-key* in **??**). So we will want to preclude the possibility of sets which fall outside of the hierarchy. Accordingly, we must add a new axiom, which ensures that every set occurs somewhere in the hierarchy.

Since the V_αs are our stages, we might simply consider adding the following as an axiom:

Regularity. $\forall A \exists \alpha\, A \subseteq V_\alpha$

This would be a perfectly reasonable approach. However, for reasons that will be explained in the next section, we will instead adopt an alternative axiom:

Axiom (Foundation). $(\forall A \neq \emptyset)(\exists B \in A) A \cap B = \emptyset$.

With some effort, we can show (in $\mathbf{ZF}^-$) that Foundation entails Regularity:

Definition spine.1. For each set A, let:

$$\begin{aligned} \mathrm{cl}_0(A) &= A, \\ \mathrm{cl}_{n+1}(A) &= \bigcup \mathrm{cl}_n(A), \\ \mathrm{trcl}(A) &= \bigcup_{n<\omega} \mathrm{cl}_n(A). \end{aligned}$$

We call $\mathrm{trcl}(A)$ the *transitive closure* of A.

The name "transitive closure" is apt:

sth:spine:foundation:subsetoftrcl **Proposition spine.2.** *$A \subseteq \mathrm{trcl}(A)$ and $\mathrm{trcl}(A)$ is a transitive set.*

Proof. Evidently $A = \mathrm{cl}_0(A) \subseteq \mathrm{trcl}(A)$. And if $x \in b \in \mathrm{trcl}(A)$, then $b \in \mathrm{cl}_n(A)$ for some n, so $x \in \mathrm{cl}_{n+1}(A) \subseteq \mathrm{trcl}(A)$. □

sth:spine:foundation:lem:TransitiveWellFounded **Lemma spine.3.** *If A is a transitive set, then there is some α such that $A \subseteq V_\alpha$.*

Proof. Recalling the definition of "lsub(X)" from **??**, define two sets:

$$D = \{x \in A : \forall \delta\ x \nsubseteq V_\delta\}$$
$$\alpha = \text{lsub}\{\delta : (\exists x \in A)(x \subseteq V_\delta \land (\forall \gamma \in \delta) x \nsubseteq V_\gamma)\}$$

Suppose $D = \emptyset$. So if $x \in A$, then there is some δ such that $x \subseteq V_\delta$ and, by the well-ordering of the ordinals, $(\forall \gamma \in \delta) x \nsubseteq V_\gamma$; hence $\delta \in \alpha$ and so $x \in V_\alpha$ by **??**. Hence $A \subseteq V_\alpha$, as required.

So it suffices to show that $D = \emptyset$. For reductio, suppose otherwise. By Foundation, there is some $B \in D \subseteq A$ such that $D \cap B = \emptyset$. If $x \in B$ then $x \in A$, since A is transitive, and since $x \notin D$, it follows that $\exists \delta\ x \subseteq V_\delta$. So now let

$$\beta = \text{lsub}\{\delta : (\exists x \in b)(x \subseteq V_\delta \land (\forall \gamma < \delta) x \nsubseteq V_\gamma)\}.$$

As before, $B \subseteq V_\beta$, contradicting the claim that $B \in D$. □

Theorem spine.4. *Regularity holds.* *sth:spine:foundation:zfentailsregularity*

Proof. Fix A; now $A \subseteq \text{trcl}(A)$ by Proposition spine.2, which is transitive. So there is some α such that $A \subseteq \text{trcl}(A) \subseteq V_\alpha$ by Lemma spine.3 □

These results show that $\mathbf{ZF}^-$ proves the conditional *Foundation* $\Rightarrow$ *Regularity*. In **??**, we will show that $\mathbf{ZF}^-$ proves *Regularity* $\Rightarrow$ *Foundation*. As such, Foundation and Regularity are *equivalent* (modulo $\mathbf{ZF}^-$). But this means that, given $\mathbf{ZF}^-$, we can justify Foundation by noting that it is equivalent to Regularity. And we can justify Regularity immediately on the basis of *Stages-are-key*.

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Bibliography