

replacement.1 The Strength of Replacement

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sec

We begin with a simple observation about the strength of Replacement: unless we go beyond \mathbf{Z} , we cannot prove the existence of any von Neumann ordinal greater than or equal to $\omega + \omega$.

Here is a sketch of why. Working in \mathbf{ZF} , consider the set $V_{\omega+\omega}$. This set acts as the domain for a *model* for \mathbf{Z} . To see this, we introduce some notation for the *relativization* of a formula:

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Definition replacement.1. For any set M , and any formula φ , let φ^M be the formula which results by restricting all of φ 's quantifiers to M . That is, replace “ $\exists x$ ” with “ $(\exists x \in M)$ ”, and replace “ $\forall x$ ” with “ $(\forall x \in M)$ ”.

It can be shown that, for every axiom φ of \mathbf{Z} , we have that $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$. But $\omega + \omega$ is not *in* $V_{\omega+\omega}$, by ???. So \mathbf{Z} is consistent with the non-existence of $\omega + \omega$.

This is why we said, in ??, that ??? cannot be proved without Replacement. For it is easy, within \mathbf{Z} , to define an explicit well-ordering which intuitively *should* have order-type $\omega + \omega$. Indeed, we gave an informal example of this in ??, when we presented the ordering on the natural numbers given by:

$$n \prec m \text{ iff either } n < m \text{ and } m - n \text{ is even,} \\ \text{or } n \text{ is even and } m \text{ is odd.}$$

But if $\omega + \omega$ does not exist, this well-ordering is not isomorphic to any ordinal. So \mathbf{Z} does *not* prove ???.

Flipping things around: Replacement allows us to prove the existence of $\omega + \omega$, and hence must allow us to prove the existence of $V_{\omega+\omega}$. And not just that. For *any* well-ordering we can define, ??? tells us that there is some α isomorphic with that well-ordering, and hence that V_α exists. In a straightforward way, then, Replacement guarantees that the hierarchy of sets must be *very tall*.

Over the next few sections, and then again in ??, we'll get a better sense of better just *how* tall Replacement forces the hierarchy to be. The simple point, for now, is that Replacement really *does* stand in need of justification!

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Bibliography