replacement.1 The Strength of Replacement

sth:replacement:strength:sec

Replacement is the axiom which makes the difference between **ZF** and **Z**. We helped ourselves to it throughout **??**–**??**. In this chapter, we will finally consider the question: is Replacement justified? To make the question sharp, it is worth observing that Replacement is really rather *strong*.

Unless we go beyond **Z**, we cannot prove the existence of any von Neumann ordinal greater than or equal to $\omega + \omega$. Here is a sketch of why. Working in **ZF**, consider the set $V_{\omega+\omega}$. We know from **??** that $\text{rank}(\omega + \omega) = V_{\omega+\omega}$. Now, this set acts as the domain for a *model* for **Z**. Indeed, where φ is any axiom of **Z**, let $\varphi^{V_{\omega+\omega}}$ be the formula which results by restricting all of φ's quantifiers to $V_{\omega+\omega}$ (that is, replace "$\exists x$" with "$(\exists x \in V_{\omega+\omega})$", and replace "$\forall x$" with "$(\forall x \in V_{\omega+\omega})$"). It can be shown that, for every axiom φ of **Z**, we have that $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$. But $\omega + \omega$ is not *in* $V_{\omega+\omega}$. So **Z** is consistent with the non-existence of $\omega + \omega$.

This is why we said, in **??**, that **??** cannot be proved without Replacement. For it is easy, within **Z**, to define an explicit well-ordering which intuitively *should* have order-type $\omega + \omega$. Indeed, we gave an informal example of this in **??**, when we presented the ordering on the natural numbers given by:

$$\begin{aligned} n \lessdot m \text{ iff either } &|n - m| \text{ is even and } n < m,\\ &\text{or } n \text{ is even and } m \text{ is odd.}\end{aligned}$$

But if $\omega + \omega$ does not exist, this well-ordering is not isomorphic to any ordinal. So **Z** does *not* prove **??**.

Flipping things around: Replacement allows us to prove the existence of $\omega + \omega$, and hence must allow us to prove the existence of $V_{\omega+\omega}$. And not just that. For *any* well-ordering we can define, **??** tells us that there is some α isomorphic with that well-ordering, and hence that V_α exists. In a straightforward way, then, Replacement guarantees that the hierarchy of sets must be *very tall*.

Over the next few sections, and then again in **??**, we'll get a better sense of better just *how* tall Replacement forces the hierarchy to be. The simple point, for now, is that Replacement really *does* stand in need of justification!

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Bibliography