replacement.1 Appendix: Results surrounding Replacement

sth:replacement:refproofs: In this section, we will prove Reflection within **ZF**. We will also prove a sense in which Reflection is equivalent to Replacement. And we will prove an interesting consequence of all this, concerning the strength of Reflection/Replacement. Warning: this is easily the most advanced bit of mathematics in this textbook.

We'll start with a lemma which, for brevity, employs the notational device of *overlining* to deal with sequences of variables or objects. So: " \bar{a}_k " abbreviates " a_{k_1}, \ldots, a_{k_n} ", where n is determined by context.

sth:replacement:refproofs: Lemma replacement.1. For each $1 \leq i \leq k$, let $\varphi_i(\overline{v}_i, x)$ be a formula. lemreflection Then for each α there is some $\beta > \alpha$ such that, for any $\overline{a}_1, \ldots, \overline{a}_k \in V_\beta$ and each $1 \leq i \leq k$:

$$\exists x \varphi_i(\overline{a}_i, x) \to (\exists x \in V_\beta) \varphi_i(\overline{a}_i, x)$$

Proof. We define a term μ as follows: $\mu(\overline{a}_1, \ldots, \overline{a}_k)$ is the least stage, V, which satisfies all of the following conditionals, for $1 \leq i \leq k$:

$$\exists x \varphi_i(\overline{a}_i, x) \to (\exists x \in V) \varphi_i(\overline{a}_i, x))$$

It is easy to confirm that $\mu(\overline{a}_1, \ldots, \overline{a}_k)$ exists for all $\overline{a}_1, \ldots, \overline{a}_k$. Now, using Replacement and our recursion theorem, define:

$$S_0 = V_{\alpha+1}$$

$$S_{n+1} = S_n \cup \bigcup \{ \mu(\overline{a}_1, \dots, \overline{a}_k) : \overline{a}_1, \dots, \overline{a}_k \in S_n \}$$

$$S = \bigcup_{m < \omega} S_n.$$

Each S_n , and hence S itself, is a stage after V_{α} . Now fix $\overline{a}_1, \ldots, \overline{a}_k \in S$; so there is some $n < \omega$ such that $\overline{a}_1, \ldots, \overline{a}_k \in S_n$. Fix some $1 \leq i \leq k$, and suppose that $\exists x \varphi_i(\overline{a}_i, x)$. So $(\exists x \in \mu(\overline{a}_1, \ldots, \overline{a}_k)) \varphi_i(\overline{a}_i, x)$ by construction, so $(\exists x \in S_{n+1}) \varphi_i(\overline{a}_i, x)$ and hence $(\exists x \in S) \varphi_i(\overline{a}_i, x)$. So S is our V_{β} . \Box

We can now prove ?? quite straightforwardly:

Proof. Fix α . Without loss of generality, we can assume φ 's only connectives are \exists , \neg and \land (since these are expressively adequate). Let ψ_1, \ldots, ψ_k enumerate each of φ 's subformulas according to complexity, so that $\psi_k = \varphi$. By Lemma replacement.1, there is a $\beta > \alpha$ such that, for any $\overline{a}_i \in V_\beta$ and each $1 \leq i \leq k$:

$$\exists x \psi_i(\overline{a}_i, x) \to (\exists x \in V_\beta) \psi_i(\overline{a}_i, x) \tag{(*)}$$

By induction on complexity of ψ_i , we will show that $\psi_i(\overline{a}_i) \leftrightarrow \psi_i^{V_\beta}(\overline{a}_i)$, for any $\overline{a}_i \in V_\beta$. If ψ_i is atomic, this is trivial. The biconditional also establishes that, when ψ_i is a negation or conjunction of subformulas satisfying this

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property, ψ_i itself satisfies this property. So the only interesting case concerns quantification. Fix $\overline{a}_i \in V_\beta$; then:

$$(\exists x \psi_i(\overline{a}_i, x))^{V_\beta} \text{ iff } (\exists x \in V_\beta) \psi_i^{V_\beta}(\overline{a}_i, x) \qquad \text{by definition} \\ \text{iff } (\exists x \in V_\beta) \psi_i(\overline{a}_i, x) \qquad \text{by hypothesis} \\ \text{iff } \exists x \psi_i(\overline{a}_i, x) \qquad \text{by } (*) \end{cases}$$

This completes the induction; the result follows as $\psi_k = \varphi$.

We have proved Reflection in **ZF**. Our proof essentially followed Montague (1961). We now want to prove in **Z** that Reflection entails Replacement. The proof follows Lévy (1960), but with a simplification.

Since we are working in \mathbf{Z} , we cannot present Reflection in exactly the form given above. After all, we formulated Reflection using the " V_{α} " notation, and that cannot be defined in \mathbf{Z} (see ??). So instead we will offer an apparently weaker formulation of Replacement, as follows:

Weak-Reflection. For any formula φ , there is a transitive set S such that 0, 1, and any parameters to φ are elements of S, and $(\forall \overline{x} \in S)(\varphi \leftrightarrow \varphi^S)$.

To use this to prove Replacement, we will first follow Lévy (1960, first part of Theorem 2) and show that we can "reflect" two formulas at once:

Lemma replacement.2 (in Z + Weak-Reflection.). For any formulas ψ , χ , sth:replacement:refproofs: there is a transitive set S such that 0 and 1 (and any parameters to the formu-lem:reflect las) are elements of S, and $(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \land (\chi \leftrightarrow \chi^S))$.

Proof. Let φ be the formula $(z = 0 \land \psi) \lor (z = 1 \land \chi)$.

Here we use an abbreviation; we should spell out "z = 0" as " $\forall t \notin z$ " and "z = 1" as " $\forall s (s \in z \leftrightarrow \forall t t \notin s)$ ". But since $0, 1 \in S$ and S is transitive, these formulas are *absolute* for S; that is, they will apply to the same object whether we restrict their quantifiers to S.¹

By Weak-Reflection, we have some appropriate S such that:

$$(\forall z, \overline{x} \in S)(\varphi \leftrightarrow \varphi^{S})$$

i.e. $(\forall z, \overline{x} \in S)(((z = 0 \land \psi) \lor (z = 1 \land \chi)) \leftrightarrow$
 $((z = 0 \land \psi) \lor (z = 1 \land \chi))^{S})$
i.e. $(\forall z, \overline{x} \in S)(((z = 0 \land \psi) \lor (z = 1 \land \chi)) \leftrightarrow$
 $((z = 0 \land \psi^{S}) \lor (z = 1 \land \chi^{S})))$
i.e. $(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^{S}) \land (\chi \leftrightarrow \chi^{S}))$

The second claim entails the third because "z = 0" and "z = 1" are absolute for S; the fourth claim follows since $0 \neq 1$.

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¹More formally, letting ξ be either of these formulas, $\xi(z) \leftrightarrow \xi^{S}(z)$.

We can now obtain Replacement, just by following and simplifying Lévy (1960, Theorem 6):

Theorem replacement.3 (in Z + Weak-Reflection). For any formula $\varphi(v, w)$, and any A, if $(\forall x \in A) \exists ! y \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y) \}$ exists.

Proof. Fix A such that $(\forall x \in A) \exists ! y \varphi(x, y)$, and define formulas:

$$\psi \text{ is } (\varphi(x, z) \land A = A)$$

$$\chi \text{ is } \exists y \, \varphi(x, y)$$

Using Lemma replacement.2, since A is a parameter to ψ , there is a transitive S such that $0, 1, A \in S$ (along with any other parameters), and such that:

$$(\forall x, z \in S)((\psi \leftrightarrow \psi^S) \land (\chi \leftrightarrow \chi^S))$$

So in particular:

$$\begin{aligned} (\forall x, z \in S)(\varphi(x, z) \leftrightarrow \varphi^S(x, z)) \\ (\forall x \in S)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi^S(x, y)) \end{aligned}$$

Combining these, and observing that $A \subseteq S$ since $A \in S$ and S is transitive:

$$(\forall x \in A) (\exists y \varphi(x, y) \leftrightarrow (\exists y \in S) \varphi(x, y))$$

Now $(\forall x \in A)(\exists ! y \in S)\varphi(x, y)$, because $(\forall x \in A)\exists ! y \varphi(x, y)$. Now Separation yields $\{y \in S : (\exists x \in A)\varphi(x, y)\} = \{y : (\exists x \in A)\varphi(x, y)\}$.

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Bibliography

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