replacement.1 Appendix: Results surrounding Replacement

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In this section, we will prove Reflection within **ZF**. We will also prove a sense in which Reflection is equivalent to Replacement. And we will prove an interesting consequence of all this, concerning the strength of Reflection/Replacement. *Warning: this is easily the most advanced bit of mathematics in this textbook.*

We'll start with a lemma which, for brevity, employs the notational device of *overlining* to deal with sequences of variables or objects. So: "$\overline{a}_k$" abbreviates "$a_{k_1}, \ldots, a_{k_n}$", where n is determined by context.

sth:replacement:refproofs:lemreflection

Lemma replacement.1. *For each $1 \leq i \leq k$, let $\varphi_i(\overline{v}_i, x)$ be a formula. Then for each α there is some $\beta > \alpha$ such that, for any $\overline{a}_1, \ldots, \overline{a}_k \in V_\beta$ and each $1 \leq i \leq k$:*

$$\exists x \varphi_i(\overline{a}_i, x) \rightarrow (\exists x \in V_\beta)\varphi_i(\overline{a}_i, x)$$

Proof. We define a term μ as follows: $\mu(\overline{a}_1, \ldots, \overline{a}_k)$ is the least stage, V, which satisfies all of the following conditionals, for $1 \leq i \leq k$:

$$\exists x \varphi_i(\overline{a}_i, x) \rightarrow (\exists x \in V)\varphi_i(\overline{a}_i, x))$$

It is easy to confirm that $\mu(\overline{a}_1, \ldots, \overline{a}_k)$ exists for all $\overline{a}_1, \ldots, \overline{a}_k$. Now, using Replacement and our recursion theorem, define:

$$\begin{aligned} S_0 &= V_{\alpha+1} \\ S_{n+1} &= S_n \cup \bigcup\{\mu(\overline{a}_1, \ldots, \overline{a}_k) : \overline{a}_1, \ldots, \overline{a}_k \in S_n\} \\ S &= \bigcup_{m<\omega} S_n. \end{aligned}$$

Each S_n, and hence S itself, is a stage after V_α. Now fix $\overline{a}_1, \ldots, \overline{a}_k \in S$; so there is some $n < \omega$ such that $\overline{a}_1, \ldots, \overline{a}_k \in S_n$. Fix some $1 \leq i \leq k$, and suppose that $\exists x \varphi_i(\overline{a}_i, x)$. So $(\exists x \in \mu(\overline{a}_1, \ldots, \overline{a}_k))\varphi_i(\overline{a}_i, x)$ by construction, so $(\exists x \in S_{n+1})\varphi_i(\overline{a}_i, x)$ and hence $(\exists x \in S)\varphi_i(\overline{a}_i, x)$. So S is our V_β. □

We can now prove **??** quite straightforwardly:

Proof. Fix α. Without loss of generality, we can assume φ's only connectives are $\exists$, $\neg$ and $\wedge$ (since these are expressively adequate). Let $\psi_1, \ldots, \psi_k$ enumerate each of φ's subformulas according to complexity, so that $\psi_k = \varphi$. By Lemma replacement.1, there is a $\beta > \alpha$ such that, for any $\overline{a}_i \in V_\beta$ and each $1 \leq i \leq k$:

$$\exists x \psi_i(\overline{a}_i, x) \rightarrow (\exists x \in V_\beta)\psi_i(\overline{a}_i, x) \tag{*}$$

By induction on complexity of ψ_i, we will show that $\psi_i(\overline{a}_i) \leftrightarrow \psi_i^{V_\beta}(\overline{a}_i)$, for any $\overline{a}_i \in V_\beta$. If ψ_i is atomic, this is trivial. The biconditional also establishes that, when ψ_i is a negation or conjunction of subformulas satisfying this property, ψ_i itself satisfies this property. So the only interesting case concerns quantification. Fix $\overline{a}_i \in V_\beta$; then:

$$\begin{aligned}
(\exists x \psi_i(\overline{a}_i, x))^{V_\beta} &\text{ iff } (\exists x \in V_\beta)\psi_i^{V_\beta}(\overline{a}_i, x) && \text{by definition}\\
&\text{ iff } (\exists x \in V_\beta)\psi_i(\overline{a}_i, x) && \text{by hypothesis}\\
&\text{ iff } \exists x \psi_i(\overline{a}_i, x) && \text{by } (*)
\end{aligned}$$

This completes the induction; the result follows as $\psi_k = \varphi$. □

We have proved Reflection in **ZF**. Our proof essentially followed Montague (1961). We now want to prove in **Z** that Reflection entails Replacement. The proof follows Lévy (1960), but with a simplification.

Since we are working in **Z**, we cannot present Reflection in exactly the form given above. After all, we formulated Reflection using the "V_α" notation, and that cannot be defined in **Z** (see **??**). So instead we will offer an apparently weaker formulation of Replacement, as follows:

> *Weak-Reflection.* For any formula φ, there is a transitive set S such that 0, 1, and any parameters to φ are elements of S, and $(\forall \overline{x} \in S)(\varphi \leftrightarrow \varphi^S)$.

To use this to prove Replacement, we will first follow Lévy (1960, first part of Theorem 2) and show that we can "reflect" two formulas at once:

Lemma replacement.2 (in $\mathbf{Z}$ + Weak-Reflection.). *For any formulas ψ, χ, there is a transitive set S such that 0 and 1 (and any parameters to the formulas) are elements of S, and $(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$.* sth:replacement:refproofs: lem:reflect

Proof. Let φ be the formula $(z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)$.

Here we use an abbreviation; we should spell out "$z = 0$" as "$\forall t\, t \notin z$" and "$z = 1$" as "$\forall s(s \in z \leftrightarrow \forall t\, t \notin s)$". But since $0, 1 \in S$ and S is transitive, these formulas are *absolute* for S; that is, they will apply to the same object whether we restrict their quantifiers to S.[1]

By Weak-Reflection, we have some appropriate S such that:

$$\begin{aligned}
&(\forall z, \overline{x} \in S)(\varphi \leftrightarrow \varphi^S)\\
\text{i.e. } &(\forall z, \overline{x} \in S)(((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)) \leftrightarrow\\
&\qquad ((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi))^S)\\
\text{i.e. } &(\forall z, \overline{x} \in S)(((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)) \leftrightarrow\\
&\qquad ((z = 0 \wedge \psi^S) \vee (z = 1 \wedge \chi^S)))\\
\text{i.e. } &(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))
\end{aligned}$$

The second claim entails the third because "$z = 0$" and "$z = 1$" are absolute for S; the fourth claim follows since $0 \neq 1$. □

[1] More formally, letting ξ be either of these formulas, $\xi(z) \leftrightarrow \xi^S(z)$.

We can now obtain Replacement, just by following and simplifying Lévy (1960, Theorem 6):

Theorem replacement.3 (in Z + Weak-Reflection). *For any formula $\varphi(v, w)$, and any A, if $(\forall x \in A)\exists! y\, \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y\}$ exists.*

Proof. Fix A such that $(\forall x \in A)\exists! y\, \varphi(x, y)$, and define formulas:

$$\psi \text{ is } (\varphi(x, z) \wedge A = A)$$
$$\chi \text{ is } \exists y\, \varphi(x, y)$$

Using Lemma replacement.2, since A is a parameter to ψ, there is a transitive S such that $0, 1, A \in S$ (along with any other parameters), and such that:

$$(\forall x, z \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$$

So in particular:

$$(\forall x, z \in S)(\varphi(x, z) \leftrightarrow \varphi^S(x, z))$$
$$(\forall x \in S)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi^S(x, y))$$

Combining these, and observing that $A \subseteq S$ since $A \in S$ and S is transitive:

$$(\forall x \in A)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi(x, y))$$

Now $(\forall x \in A)(\exists! y \in S)\varphi(x, y)$, because $(\forall x \in A)\exists! y\, \varphi(x, y)$. Now Separation yields $\{y \in S : (\exists x \in A)\varphi(x, y)\} = \{y : (\exists x \in A)\varphi(x, y)\}$. □

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