replacement.1  Appendix: Results surrounding Replacement

In this section, we will prove Reflection within ZF. We will also prove a sense in which Reflection is equivalent to Replacement. And we will prove an interesting consequence of all this, concerning the strength of Reflection/Replacement.

Warning: this is easily the most advanced bit of mathematics in this textbook.

We'll start with a lemma which, for brevity, employs the notational device of overlining to deal with sequences of variables or objects. So: “\( \overline{a_k, \ldots, a_n} \)” abbreviates “\( a_{k_1}, \ldots, a_{n_1} \)” , where \( n \) is determined by context.

Lemma replacement.1. For each \( 1 \leq i \leq k \), let \( \varphi_i(\overline{a_i}, x) \) be a formula. Then for each \( \alpha \) there is some \( \beta > \alpha \) such that, for any \( \overline{a_1}, \ldots, \overline{a_k} \in V_\beta \) and each \( 1 \leq i \leq k \):

\[
\exists x \varphi_i(\overline{a_i}, x) \rightarrow (\exists x \in V_\beta) \varphi_i(\overline{a_i}, x)
\]

Proof. We define a term \( \mu \) as follows: \( \mu(\overline{a_1}, \ldots, \overline{a_k}) \) is the least stage, \( V_\beta \), which satisfies all of the following conditionals, for \( 1 \leq i \leq k \):

\[
\exists x \varphi_i(\overline{a_i}, x) \rightarrow (\exists x \in V_\beta) \varphi_i(\overline{a_i}, x)
\]

It is easy to confirm that \( \mu(\overline{a_1}, \ldots, \overline{a_k}) \) exists for all \( \overline{a_1}, \ldots, \overline{a_k} \). Now, using Replacement and our recursion theorem, define:

\[
S_0 = V_{\alpha+1}
\]

\[
S_{n+1} = S_n \cup \bigcup \{ \mu(\overline{a_1}, \ldots, \overline{a_k}) : \overline{a_1}, \ldots, \overline{a_k} \in S_n \}
\]

\[
S = \bigcup_{m<\omega} S_m.
\]

Each \( S_n \), and hence \( S \) itself, is a stage after \( V_\alpha \). Now fix \( \overline{a_1}, \ldots, \overline{a_k} \in S \); so there is some \( n < \omega \) such that \( \overline{a_1}, \ldots, \overline{a_k} \in S_n \). Fix some \( 1 \leq i \leq k \), and suppose that \( \exists x \varphi_i(\overline{a_i}, x) \). So \( (\exists x \in \mu(\overline{a_1}, \ldots, \overline{a_k})) \varphi_i(\overline{a_i}, x) \) by construction, so \( (\exists x \in S_{n+1}) \varphi_i(\overline{a_i}, x) \) and hence \( (\exists x \in S) \varphi_i(\overline{a_i}, x) \). So \( S \) is our \( V_\beta \). □

We can now prove ?? quite straightforwardly:

Proof. Fix \( \alpha \). Without loss of generality, we can assume \( \varphi \)'s only connectives are \( \exists, \neg \) and \( \wedge \) (since these are expressively adequate). Let \( \psi_1, \ldots, \psi_k \) enumerate each of \( \varphi \)'s subformulas according to complexity, so that \( \psi_k = \varphi \). By Lemma replacement.1, there is a \( \beta > \alpha \) such that, for any \( \overline{a_i} \in V_\beta \) and each \( 1 \leq i \leq k \):

\[
\exists x \psi_i(\overline{a_i}, x) \rightarrow (\exists x \in V_\beta) \psi_i(\overline{a_i}, x)
\]

By induction on complexity of \( \psi_i \), we will show that \( \psi_i(\overline{a_i}) \leftrightarrow \psi_i^{V_\beta}(\overline{a_i}) \), for any \( \overline{a_i} \in V_\beta \). If \( \psi_i \) is atomic, this is trivial. The biconditional also establishes that, when \( \psi_i \) is a negation or conjunction of subformulas satisfying this
property, \(\psi_i\) itself satisfies this property. So the only interesting case concerns quantification. Fix \(\pi_i \in V_\beta\); then:

\[
(\exists x \psi_i(\pi_i, x))^{V_\beta} \iff (\exists x \in V_\beta)\psi_i(\pi_i, x)
\]

by definition

\[
(\exists x \in V_\beta)\psi_i(\pi_i, x)
\]

by hypothesis

\[
\exists x \psi_i(\pi_i, x)
\]

by (*)

This completes the induction; the result follows as \(\psi_k = \varphi\).

We have proved Reflection in \(ZF\). Our proof essentially followed Montague (1961). We now want to prove in \(Z\) that Reflection entails Replacement. The proof follows Lévy (1960), but with a simplification.

Since we are working in \(Z\), we cannot present Reflection in exactly the form given above. After all, we formulated Reflection using the “\(V_\alpha\)” notation, and that cannot be defined in \(Z\) (see ??). So instead we will offer an apparently weaker formulation of Replacement, as follows:

**Weak-Reflection.** For any formula \(\varphi\), there is a transitive set \(S\) such that 0, 1, and any parameters to \(\varphi\) are elements of \(S\), and \((\forall \pi \in S)(\varphi \leftrightarrow \varphi^S)\).

To use this to prove Replacement, we will first follow Lévy (1960, first part of Theorem 2) and show that we can “reflect” two formulas at once:

**Lemma replacement.2 (in \(Z + Weak-Reflection\).)** For any formulas \(\psi, \chi\), there is a transitive set \(S\) such that 0 and 1 (and any parameters to the formulas) are elements of \(S\), and \((\forall \pi \in S)((\psi \leftrightarrow \psi^S) \land (\chi \leftrightarrow \chi^S))\).

**Proof.** Let \(\varphi\) be the formula \((z = 0 \land \psi) \lor (z = 1 \land \chi)\).

Here we use an abbreviation; we should spell out “\(z = 0\)” as “\(\forall t t \notin z\)” and “\(z = 1\)” as “\(\forall s (s \in z \leftrightarrow \forall t t \notin s)\)”. But since 0, 1 \(\in S\) and \(S\) is transitive, these formulas are absolute for \(S\); that is, they will apply to the same object whether we restrict their quantifiers to \(S\).

By Weak-Reflection, we have some appropriate \(S\) such that:

\[
(\forall z, \pi \in S)(\varphi \leftrightarrow \varphi^S)
\]

i.e. \((\forall z, \pi \in S)(((z = 0 \land \psi) \lor (z = 1 \land \chi)) \leftrightarrow ((z = 0 \land \psi^S) \lor (z = 1 \land \chi^S)))\)

i.e. \((\forall z, \pi \in S)((z = 0 \land \psi^S) \lor (z = 1 \land \chi^S))\)

i.e. \((\forall \pi \in S)((\psi \leftrightarrow \psi^S) \land (\chi \leftrightarrow \chi^S))\)

The second claim entails the third because “\(z = 0\)” and “\(z = 1\)” are absolute for \(S\); the fourth claim follows since 0 \(\neq 1\).
We can now obtain Replacement, just by following and simplifying Lévy (1960, Theorem 6):

**Theorem replacement.3 (in $\mathbb{Z} + \text{Weak-Reflection}$).** For any formula $\varphi(v, w)$, and any $A$, if $(\forall x \in A)\exists ! y \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y)\}$ exists.

**Proof.** Fix $A$ such that $(\forall x \in A)\exists ! y \varphi(x, y)$, and define formulas:

- $\psi$ is $(\varphi(x, z) \land A = A)$
- $\chi$ is $\exists y \varphi(x, y)$

Using Lemma replacement.2, since $A$ is a parameter to $\psi$, there is a transitive $S$ such that $0, 1, A \in S$ (along with any other parameters), and such that:

$$(\forall x, z \in S)((\psi \leftrightarrow \psi^S) \land (\chi \leftrightarrow \chi^S))$$

So in particular:

$$(\forall x, z \in S)(\varphi(x, z) \leftrightarrow \varphi^S(x, z))$$

$$(\forall x \in S)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi^S(x, y))$$

Combining these, and observing that $A \subseteq S$ since $A \in S$ and $S$ is transitive:

$$(\forall x \in A)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi(x, y))$$

Now $(\forall x \in A)(\exists y \varphi(x, y))$, because $(\forall x \in A)\exists y \varphi(x, y)$. Now Separation yields $\{y \in S : (\exists x \in A)\varphi(x, y)\} = \{y : (\exists x \in A)\varphi(x, y)\}$. \qed

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**Bibliography**
