replacement.1 Appendix: Finite axiomatizability

^{sec} We close this chapter by extracting some results from Replacement. The first result is due to Montague (1961); note that it is not a proof within **ZF**, but a proof about **ZF**:

eplacement:finiteaxiomatizability: Theorem replacement.1. ZF is not finitely axiomatizable. More generally: $z_{fnotfinitely}$ if T is finite and $T \vdash ZF$, then T is inconsistent.

> (Here, we tacitly restrict ourselves to first-order sentences whose only nonlogical primitive is \in , and we write $\mathbf{T} \vdash \mathbf{ZF}$ to indicate that $\mathbf{T} \vdash \varphi$ for all $\varphi \in \mathbf{ZF}$.)

> *Proof.* Fix finite **T** such that $\mathbf{T} \vdash \mathbf{ZF}$. So, **T** proves Reflection, i.e. ??. Since **T** is finite, we can rewrite it as a single conjunction, θ . Reflecting with this formula, $\mathbf{T} \vdash \exists \beta (\theta \leftrightarrow \theta^{V_{\beta}})$. Since trivially $\mathbf{T} \vdash \theta$, we find that $\mathbf{T} \vdash \exists \beta \ \theta^{V_{\beta}}$.

Now, let $\psi(X)$ abbreviate:

$$\theta^X \wedge X$$
 is transitive $\wedge (\forall Y \in X)(Y \text{ is transitive} \rightarrow \neg \theta^Y)$

roughly this says: X is a transitive model of θ , and \in -minimal in this regard. Now, recalling that $\mathbf{T} \vdash \exists \beta \ \theta^{V_{\beta}}$, by basic facts about ranks within **ZF** and hence within **T**, we have:

$$\mathbf{T} \vdash \exists M \psi(M). \tag{*}$$

Using the first conjunct of $\psi(X)$, whenever $\mathbf{T} \vdash \sigma$, we have that $\mathbf{T} \vdash \forall X(\psi(X) \rightarrow \sigma^X)$. So, by (*):

 $\mathbf{T} \vdash \forall X(\psi(X) \to (\exists N\psi(N))^X)$

Using this, and (*) again:

$$\mathbf{T} \vdash \exists M(\psi(M) \land (\exists N\psi(N))^M)$$

In particular, then:

$$\mathbf{T} \vdash \exists M(\psi(M) \land (\exists N \in M)((N \text{ is transitive})^N \land (\theta^N)^M))$$

So, by elementary reasoning concerning transitivity:

$$\mathbf{T} \vdash \exists M(\psi(M) \land (\exists N \in M)(N \text{ is transitive} \land \theta^N))$$

So that \mathbf{T} is inconsistent.¹

Here is a similar result, noted by Potter (2004, 223):

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 $^{^1\}mathrm{This}$ "elementary reasoning" involves proving certain "absoluteness facts" for transitive sets.

finite extension of Z

Proposition replacement.2. Let \mathbf{T} extend \mathbf{Z} with finitely many new ax- sth:replacement:finiteaxiomatizab ioms. If $\mathbf{T} \vdash \mathbf{ZF}$, then \mathbf{T} is inconsistent. (Here we use the same tacit restrictions as for Theorem replacement.1.)

Proof. Use θ for the conjunction of all of **T**'s axioms *except* for the (infinitely many) instances of Separation. Defining ψ from θ as in Theorem replacement.1, we can show that $\mathbf{T} \vdash \exists M \psi(M)$.

As in Theorem replacement.1, we can establish the schema that, whenever $\mathbf{T} \vdash \sigma$, we have that $\mathbf{T} \vdash \forall X(\psi(X) \rightarrow \sigma^X)$. We then finish our proof, exactly as in Theorem replacement.1.

However, establishing the schema involves a little more work than in Theorem replacement.1. After all, the Separation-instances are in **T**, but they are not conjuncts of θ . However, we can overcome this obstacle by proving that $\mathbf{T} \vdash \forall X(X \text{ is transitive} \rightarrow \sigma^X)$, for every Separation-instance σ . We leave this to the reader. \Box

Problem replacement.1. Show that, for every Separation-instance σ , we have: $\mathbf{Z} \vdash \forall X(X \text{ is transitive} \rightarrow \sigma^X)$. (We used this schema in Proposition replacement.2.)

Problem replacement.2. Show that, for every $\varphi \in \mathbf{Z}$, we have $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$.

Problem replacement.3. Confirm the remaining schematic results invoked in the proofs of Theorem replacement.1 and Proposition replacement.2.

As remarked in ??, this shows that Replacement is strictly stronger than ??. Or, slightly more strictly: if \mathbf{Z} + "every well-ordering is isomorphic to a unique ordinal" is consistent, then it fails to prove some Replacement-instance.

Photo Credits

Bibliography

- Montague, Richard. 1961. Semantic closure and non-finite axiomatizability I. In Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics (Warsaw 1959), 45–69. New York: Pergamon.
- Potter, Michael. 2004. Set Theory and its Philosophy. Oxford: Oxford University Press.