ordinals.1 Well-Orderings

sth:ordinals:wo:sec

The fundamental notion is as follows:

Definition ordinals.1. The relation $<$ *well-orders* A iff it meets these two conditions:

1. $<$ is connected, i.e., for all $a, b \in A$, either $a < b$ or $a = b$ or $b < a$;
2. every non-empty subset of A has a $<$-minimal element, i.e., if $\emptyset \neq X \subseteq A$ then $(\exists m \in X)(\forall z \in X) z \not< m$

It is easy to see that three examples we just considered were indeed well-ordering relations.

Problem ordinals.1. **??** presented three example orderings on the natural numbers. Check that each is a well-ordering.

Here are some elementary but extremely important observations concerning well-ordering.

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Proposition ordinals.2. *If $<$ well-orders A, then every non-empty subset of A has a $<$-least member, and $<$ is irreflexive, asymmetric and transitive.*

Proof. If X is a non-empty subset of A, it has a $<$-minimal element m, i.e., $(\forall z \in X) z \not< m$. Since $<$ is connected, $(\forall z \in X) m \leq z$. So m is $<$-least.

For irreflexivity, fix $a \in A$; since $\{a\}$ has a $<$-least element, $a \not< a$. For transitivity, if $a < b < c$, then since $\{a, b, c\}$ has a $<$-least element, $a < c$. Asymmetry follows from irreflexivity and transitivity □

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Proposition ordinals.3. *If $<$ well-orders A, then for any formula $\varphi(x)$:*[1]

$$\textit{if } (\forall a \in A)((\forall b < a)\varphi(b) \rightarrow \varphi(a)), \textit{ then } (\forall a \in A)\varphi(a).$$

Proof. We will prove the contrapositive. Suppose $\neg(\forall a \in A)\varphi(a)$, i.e., that $X = \{x \in A : \neg\varphi(x)\} \neq \emptyset$. Then X has an $<$-minimal element, a. So $(\forall b < a)\varphi(b)$ but $\neg\varphi(a)$. □

This last property should remind you of the principle of strong induction on the naturals, i.e.: if $(\forall n \in \omega)((\forall m < n)\varphi(m) \rightarrow \varphi(n))$, then $(\forall n \in \omega)\varphi(n)$. And this property makes well-ordering into a very *robust* notion.

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Bibliography

[1] which may have parameters