ordinals.1 Replacement

sth:ordinals:replacement:sec

In **??**, we motivated the introduction of ordinals by suggesting that we could treat them as order-types, i.e., canonical proxies for well-orderings. In order for that to work, we would need to prove that *every well-ordering is isomorphic to some ordinal*. This would allow us to define $\text{ord}(A, <)$ as the ordinal α such that $\langle A, <\rangle \cong \alpha$.

Unfortunately, we *cannot* prove the desired result only the Axioms we provided introduced so far. (We will see why in **??**, but for now the point is: we can't.) We need a new thought, and here it is:

Axiom (Scheme of Replacement). For any formula $\varphi(x, y)$, the following is an axiom:

$$\text{for any } A\text{, if } (\forall x \in A)\exists! y\, \varphi(x, y)\text{, then } \{y : (\exists x \in A)\varphi(x, y)\} \text{ exists.}$$

As with Separation, this is a scheme: it yields infinitely many axioms, for each of the infinitely many different φ's. And it can equally well be (and normally is) written down thus:

For any formula $\varphi(x, y)$ which does not contain "B", the following is an axiom:

$$\forall A[(\forall x \in A)\exists! y\, \varphi(x, y) \rightarrow \exists B \forall y(y \in B \leftrightarrow (\exists x \in A)\varphi(x, y))]$$

On first encounter, however, this is quite a tangled formula. The following quick consequence of Replacement probably gives a *clearer* expression to the intuitive idea we are working with:

Corollary ordinals.1. *For any term $\tau(x)$, and any set A, this set exists:*

$$\{\tau(x) : x \in A\} = \{y : (\exists x \in A)y = \tau(x)\}.$$

Proof. Since τ is a *term*, $\forall x \exists! y\, \tau(x) = y$. A fortiori, $(\forall x \in A)\exists! y\, \tau(x) = y$. So $\{y : (\exists x \in A)\tau(x) = y\}$ exists by Replacement. □

This suggests that "Replacement" is a good name for the Axiom: given a set A, you can form a new set, $\{\tau(x) : x \in A\}$, by replacing every member of A with its image under τ. Indeed, following the notation for the image of a set under a function, we might write $\tau[A]$ for $\{\tau(x) : x \in A\}$.

Crucially, however, τ is a *term*. It need not be (a name for) a *function*, in the sense of **??**, i.e., a certain set of ordered pairs. After all, if f is a function (in that sense), then the set $f[A] = \{f(x) : x \in A\}$ is just a particular subset of $\text{ran}(f)$, and that is already guaranteed to exist, just using the axioms of $\mathbf{Z}^-$.[1]

[1] Just consider $\{y \in \bigcup\bigcup f : (\exists x \in A)y = f(x)\}$.

Replacement, by contrast, is a *powerful* addition to our axioms, as we will see in **??**.

Photo Credits

Bibliography