

ordinals.1 Ordinals as Order-Types

sth:ordinals:ordtype:
sec Armed with Replacement, and so now working in \mathbf{ZF}^- , we can finally prove the result we have been aiming for:

sth:ordinals:ordtype:
thmOrdinalRepresentation **Theorem ordinals.1.** *Every well-ordering is isomorphic to a unique ordinal.*

Proof. Let $\langle A, < \rangle$ be a well-order. By ??, it is isomorphic to at most one ordinal. So, for reductio, suppose $\langle A, < \rangle$ is not isomorphic to *any* ordinal. We will first “make $\langle A, < \rangle$ as small as possible”. In detail: if some proper initial segment $\langle A_a, <_a \rangle$ is not isomorphic to any ordinal, there is a least $a \in A$ with that property; then let $B = A_a$ and $\leq = <_a$. Otherwise, let $B = A$ and $\leq = <$.

By definition, every proper initial segment of B is isomorphic to some ordinal, which is unique as above. So by Replacement, the following set exists, and is a function:

$$f = \{\langle \beta, b \rangle : b \in B \text{ and } \beta \cong \langle B_b, \leq_b \rangle\}$$

To complete the reductio, we’ll show that f is an isomorphism $\alpha \rightarrow B$, for some ordinal α .

It is obvious that $\text{ran}(f) = B$. And by ??, f preserves ordering, i.e., $\gamma \in \beta$ iff $f(\gamma) < f(\beta)$. To show that $\text{dom}(f)$ is an ordinal, by ?? it suffices to show that $\text{dom}(f)$ is transitive. So fix $\beta \in \text{dom}(f)$, i.e., $\beta \cong \langle B_b, \leq_b \rangle$ for some b . If $\gamma \in \beta$, then $\gamma \in \text{dom}(f)$ by ??; generalising, $\beta \subseteq \text{dom}(f)$. \square

This result licenses the following definition, which we have wanted to offer since ??:

Definition ordinals.2. If $\langle A, < \rangle$ is a well-ordering, then its order type, $\text{ord}(A, < \rangle)$, is the unique ordinal α such that $\langle A, < \rangle \cong \alpha$.

Moreover, this definition licenses two nice principles:

sth:ordinals:ordtype:
ordtypesworklikeyouwant **Corollary ordinals.3.** *Where $\langle A, < \rangle$ and $\langle B, \leq \rangle$ are well-orderings:*

$$\begin{aligned} \text{ord}(A, < \rangle) = \text{ord}(B, \leq \rangle) &\text{ iff } \langle A, < \rangle \cong \langle B, \leq \rangle \\ \text{ord}(A, < \rangle) \in \text{ord}(B, \leq \rangle) &\text{ iff } \langle A, < \rangle \cong \langle B_b, \leq_b \rangle \text{ for some } b \in B \end{aligned}$$

Proof. The identity holds by ??. To prove the second claim, let $\text{ord}(A, < \rangle) = \alpha$ and $\text{ord}(B, \leq \rangle) = \beta$, and let $f: \beta \rightarrow \langle B, \leq \rangle$ be our isomorphism. Then:

$$\begin{aligned} \alpha \in \beta &\text{ iff } f \upharpoonright_\alpha : \alpha \rightarrow B_{f(\alpha)} \text{ is an isomorphism} \\ &\text{ iff } \langle A, < \rangle \cong \langle B_{f(\alpha)}, \leq_{f(\alpha)} \rangle \\ &\text{ iff } \langle A, < \rangle \cong \langle B_b, \leq_b \rangle \text{ for some } b \in B \end{aligned}$$

by ??, ??, and ??. \square

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Bibliography