

## ordinals.1 Successor and Limit Ordinals

sth:ordinals:opps:sec In the next few chapters, we will use ordinals a great deal. So it will help if we introduce some simple notions.

**Definition ordinals.1.** For any ordinal  $\alpha$ , its *successor* is  $\alpha^+ = \alpha \cup \{\alpha\}$ . We say that  $\alpha$  is a *successor* ordinal if  $\beta^+ = \alpha$  for some ordinal  $\beta$ . We say that  $\alpha$  is a *limit* ordinal iff  $\alpha$  is neither empty nor a successor ordinal.

The following result shows that this is the right notion of *successor*:

**Proposition ordinals.2.** *For any ordinal  $\alpha$ :*

1.  $\alpha \in \alpha^+$ ;
2.  $\alpha^+$  is an ordinal;
3. there is no ordinal  $\beta$  such that  $\alpha \in \beta \in \alpha^+$ .

*Proof.* Trivially,  $\alpha \in \alpha \cup \{\alpha\} = \alpha^+$ . Equally,  $\alpha^+$  is a transitive set of ordinals, and hence an ordinal by ???. And it is impossible that  $\alpha \in \beta \in \alpha^+$ , since then either  $\beta \in \alpha$  or  $\beta = \alpha$ , contradicting ???  $\square$

This also licenses a variant of proof by transfinite induction:

sth:ordinals:opps:simpletransrecursion **Theorem ordinals.3 (Simple Transfinite Induction).** *Let  $\varphi(x)$  be a formula such that:<sup>1</sup>*

1.  $\varphi(\emptyset)$ ; and
2. for any ordinal  $\alpha$ , if  $\varphi(\beta)$  then  $\varphi(\alpha^+)$ ; and
3. if  $\alpha$  is a limit ordinal and  $(\forall \beta \in \alpha)\varphi(\beta)$ , then  $\varphi(\alpha)$ .

Then  $\forall \alpha \varphi(\alpha)$ .

*Proof.* We prove the contrapositive. So, suppose there is some ordinal which is  $\neg\varphi$ ; let  $\gamma$  be the least such ordinal. Then either  $\gamma = \emptyset$ , or  $\gamma = \alpha^+$  for some  $\alpha$  such that  $\varphi(\alpha)$ ; or  $\gamma$  is a limit ordinal and  $(\forall \beta \in \gamma)\varphi(\beta)$ .  $\square$

A final bit of notation will prove helpful.

sth:ordinals:opps:defsupstrict **Definition ordinals.4.** If  $X$  is a set of ordinals, then  $\text{lsub}(X) = \bigcup_{\alpha \in X} \alpha^+$ .

Here, “lsub” stands for “least strict upper bound”.<sup>2</sup> The following result explains this:

<sup>1</sup>The formula may have parameters.

<sup>2</sup>Some books use “sup( $X$ )” for this. But other books use “sup( $X$ )” for the least *non-strict* upper bound, i.e., simply  $\bigcup X$ . If  $X$  has a greatest element,  $\alpha$ , these notions come apart: the least *strict* upper bound is  $\alpha^+$ , whereas the least *non-strict* upper bound is just  $\alpha$ .

**Proposition ordinals.5.** *If  $X$  is a set of ordinals,  $\text{lsub}(X)$  is the least ordinal greater than every ordinal in  $X$ .*

*Proof.* Let  $Y = \{\alpha^+ : \alpha \in X\}$ , so that  $\text{lsub}(X) = \bigcup Y$ . Since ordinals are transitive and every member of an ordinal is an ordinal,  $\text{lsub}(X)$  is a transitive set of ordinals, and so is an ordinal by ??.

If  $\alpha \in X$ , then  $\alpha^+ \in Y$ , so  $\alpha^+ \subseteq \bigcup Y = \text{lsub}(X)$ , and hence  $\alpha \in \text{lsub}(X)$ . So  $\text{lsub}(X)$  is strictly greater than every ordinal in  $X$ .

Conversely, if  $\alpha \in \text{lsub}(X)$ , then  $\alpha \in \beta^+ \in Y$  for some  $\beta \in X$ , so that  $\alpha \leq \beta \in X$ . So  $\text{lsub}(X)$  is the *least* strict upper bound on  $X$ .  $\square$

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## Bibliography