ordinals.1  Basic Properties of the Ordinals

We observed that the first few ordinals are the natural numbers. The main reason for developing a theory of ordinals is to extend the principle of induction which holds on the natural numbers. We will build up to this via a sequence of elementary results.

Lemma ordinals.1. Every element of an ordinal is an ordinal.

Proof. Let $\alpha$ be an ordinal with $b \in \alpha$. Since $\alpha$ is transitive, $b \subseteq \alpha$. So $\in$ well-orders $b$ as $\in$ well-orders $\alpha$.

To see that $b$ is transitive, suppose $x \in c \in b$. So $c \in \alpha$ as $b \subseteq \alpha$. Again, as $\alpha$ is transitive, $c \subseteq \alpha$, so that $x \in \alpha$. So $x, c, b \in \alpha$. But $\in$ well-orders $\alpha$, so that $\in$ is a transitive relation on $\alpha$ by ???.

Generalising, $c \subseteq b$.

Corollary ordinals.2. $\alpha = \{\beta \in \alpha : \beta$ is an ordinal$\}$, for any ordinal $\alpha$.

Proof. Immediate from Lemma ordinals.1.

The rough gist of the next two main results, Theorem ordinals.3 and Theorem ordinals.4, is that the ordinals themselves are well-ordered by membership:

Theorem ordinals.3 (Transfinite Induction). For any formula $\varphi(x)$:

if $\exists \alpha \varphi(\alpha)$, then $\exists \alpha (\varphi(\alpha) \land (\forall \beta \in \alpha) \neg \varphi(\beta))$

where the displayed quantifiers are implicitly restricted to ordinals.

Proof. Suppose $\varphi(\alpha)$, for some ordinal $\alpha$. If $(\forall \beta \in \alpha) \neg \varphi(\beta)$, then we are done. Otherwise, as $\alpha$ is an ordinal, it has some $\in$-least element which is $\varphi$, and this is an ordinal by Lemma ordinals.1.

Note that we can equally express Theorem ordinals.3 as the scheme:

if $\forall \alpha ((\forall \beta \in \alpha) \varphi(\beta) \rightarrow \varphi(\alpha))$, then $\forall \alpha \varphi(\alpha)$

just by taking $\neg \varphi(\alpha)$ in Theorem ordinals.3, and then performing elementary logical manipulations.

Theorem ordinals.4 (Trichotomy). $\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha$, for any ordinals $\alpha$ and $\beta$.

Proof. The proof is by double induction, i.e., using Theorem ordinals.3 twice. Say that $x$ is comparable with $y$ iff $x \in y \lor x = y \lor y \in x$.

For induction, suppose that every ordinal in $\alpha$ is comparable with every ordinal. For further induction, suppose that $\alpha$ is comparable with every ordinal in $\beta$. We will show that $\alpha$ is comparable with $\beta$. By induction on $\beta$, it will follow that $\alpha$ is comparable with every ordinal; and so by induction on $\alpha$, every
ordinal is comparable with every ordinal, as required. It suffices to assume that α /∈ β and β /∈ α, and show that α = β.

To show that α ⊆ β, fix γ ∈ α; this is an ordinal by Lemma ordinals.1. So by the first induction hypothesis, γ is comparable with β. But if either γ = β or β ∈ γ then β ∈ α (invoking the fact that α is transitive if necessary), contrary to our assumption; so γ ∈ β. Generalising, α ⊆ β.

Exactly similar reasoning, using the second induction hypothesis, shows that β ⊆ α. So α = β.

As such, we will sometimes write α < β rather than α ∈ β, since ∈ is behaving as an ordering relation. There are no deep reasons for this, beyond familiarity, and because it is easier to write α ≤ β than α ∈ β ∨ α = β.\footnote{We could write α ≤ β; but that would be wholly non-standard.}

Here are two quick consequences of our last results, the first of which puts our new notation into action:

**Corollary ordinals.5.** If ∃αφ(α), then ∃α(φ(α) ∧ ∀β(φ(β) → α ≤ β)). Moreover, for any ordinals α, β, γ, both α /∈ α and α ∈ β ∈ γ → α ∈ γ.

**Proof.** Just like ??.

**Problem ordinals.1.** Complete the “exactly similar reasoning” in the proof of Theorem ordinals.4.

**Corollary ordinals.6.** A is an ordinal iff A is a transitive set of ordinals.

**Proof.** Left-to-right. By Lemma ordinals.1. Right-to-left. If A is a transitive set of ordinals, then ∈ well-orders A by Theorem ordinals.3 and Theorem ordinals.4.

Now, we glossed Theorem ordinals.3 and Theorem ordinals.4 as telling us that ∈ well-orders the ordinals. However, we have to be very cautious about this sort of claim, thanks to the following result:

**Theorem ordinals.7 (Burali-Forti Paradox).** There is no set of all the ordinals.

**Proof.** For reductio, suppose O is the set of all ordinals. If α ∈ β ∈ O, then α is an ordinal, by Lemma ordinals.1, so α ∈ O. So O is transitive, and hence O is an ordinal by Corollary ordinals.6. Hence O ∈ O, contradicting Corollary ordinals.5.\footnote{This result is named after Burali-Forti. But, it was Cantor in 1899—in a letter to Dedekind—who first saw clearly the contradiction in supposing that there is a set of all the ordinals. As van Heijenoort explains:}
Burali-Forti himself considered the contradiction as establishing, by *reductio ad absurdum*, the result that the natural ordering of the ordinals is just a partial ordering. (Heijenoort, 1967, p. 105)

Setting Burali-Forti’s mistake to one side, we can summarize the foregoing as follows. Ordinals are sets which are individually well-ordered by membership, and collectively well-ordered by membership (without collectively constituting a set).

Rounding this off, here are some more basic properties about the ordinals which follow from Theorem ordinals.3 and Theorem ordinals.4.

**Proposition ordinals.8.** *Any strictly descending sequence of ordinals is finite.*

*Proof.* Any infinite strictly descending sequence of ordinals $\alpha_0 > \alpha_1 > \alpha_2 > \ldots$ has no $\prec$-minimal member, contradicting Theorem ordinals.3.

**Proposition ordinals.9.** $\alpha \subseteq \beta \lor \beta \subseteq \alpha$, for any ordinals $\alpha, \beta$.

*Proof.* If $\alpha \in \beta$, then $\alpha \subseteq \beta$ as $\beta$ is transitive. Similarly, if $\beta \in \alpha$, then $\beta \subseteq \alpha$. And if $\alpha = \beta$, then $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$. So by Theorem ordinals.4 we are done.

**Proposition ordinals.10.** $\alpha = \beta$ iff $\alpha \equiv \beta$, for any ordinals $\alpha, \beta$.

*Proof.* The ordinals are well-orders; so this is immediate from Trichotomy (Theorem ordinals.4) and ??.

**Problem ordinals.2.** Prove that, if every member of $X$ is an ordinal, then $\bigcup X$ is an ordinal.

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**Bibliography**
