

ord-arithmetic.1 Using Ordinal Addition

Using addition on the ordinals, we can explicitly calculate the ranks of various sets, in the sense of ??:

Lemma ord-arithmetic.1. *If $\text{rank}(A) = \alpha$ and $\text{rank}(B) = \beta$, then:*

1. $\text{rank}(\wp(A)) = \alpha + 1$
2. $\text{rank}(\{A, B\}) = \max(\alpha, \beta) + 1$
3. $\text{rank}(A \cup B) = \max(\alpha, \beta)$
4. $\text{rank}(\langle A, B \rangle) = \max(\alpha, \beta) + 2$
5. $\text{rank}(A \times B) \leq \max(\alpha, \beta) + 2$
6. $\text{rank}(\bigcup A) = \alpha$ when α is empty or a limit; $\text{rank}(\bigcup A) = \gamma$ when $\alpha = \gamma + 1$

Proof. Throughout, we invoke ?? repeatedly.

(1). If $x \subseteq A$ then $\text{rank}(x) \leq \text{rank}(A)$. So $\text{rank}(\wp(A)) \leq \alpha + 1$. Since $A \in \wp(A)$ in particular, $\text{rank}(\wp(A)) = \alpha + 1$.

(2). By ??

(3). By ??.

(4). By (2), twice.

(5). Note that $A \times B \subseteq \wp(\wp(A \cup B))$, and invoke (4).

(6). If $\alpha = \gamma + 1$, there is some $c \in A$ with $\text{rank}(c) = \gamma$, and no element of A has higher rank; so $\text{rank}(\bigcup A) = \gamma$. If α is a limit ordinal, then A has elements with rank arbitrarily close to (but strictly less than) α , so that $\bigcup A$ also has elements with rank arbitrarily close to (but strictly less than) α , so that $\text{rank}(\bigcup A) = \alpha$. \square

We leave it as an exercise to show why (5) involves an inequality.

Problem ord-arithmetic.1. Produce sets A and B such that $\text{rank}(A \times B) = \max(\text{rank}(A), \text{rank}(B))$. Produce sets A and B such that $\text{rank}(A \times B) > \max(\text{rank}(A), \text{rank}(B))$.
2. Are any other ranks possible?

We are also now in a position to prove that several reasonable notions of “finite” coincide, when considering ordinals:

Lemma ord-arithmetic.2. *For any ordinal α , the following are equivalent:*

1. $\alpha \notin \omega$, i.e., α is not a natural number
2. $\omega \leq \alpha$
3. $1 + \alpha = \alpha$
4. $\alpha \approx \alpha + 1$, i.e., α and $\alpha + 1$ are equinumerous

5. α is Dedekind infinite

*sth.ord-arithmetic:using-addition.
ord:infinite*

Proof. (1) \Rightarrow (2). By Trichotomy.

(2) \Rightarrow (3). Fix $\alpha \geq \omega$. By Transfinite Induction, there is some least ordinal γ (possibly 0) such that there is a limit ordinal β with $\alpha = \beta + \gamma$. Now:

$$1 + \alpha = 1 + (\beta + \gamma) = (1 + \beta) + \gamma = \text{lsub}_{1+\delta}(\delta < \beta) + \gamma = \beta + \gamma = \alpha.$$

(3) \Rightarrow (4). There is clearly a bijection $f: (\alpha \sqcup 1) \rightarrow (1 \sqcup \alpha)$. If $1 + \alpha = \alpha$, there is an isomorphism $g: (1 \sqcup \alpha) \rightarrow \alpha$. Now consider $g \circ f$.

(4) \Rightarrow (5). If $\alpha \approx \alpha + 1$, there is a bijection $f: (\alpha \sqcup 1) \rightarrow \alpha$. Define $g(\gamma) = f(\gamma, 0)$ for each $\gamma < \alpha$; this injection witnesses that α is Dedekind infinite, since $f(0, 1) \in \alpha \setminus \text{ran}(g)$.

(5) \Rightarrow (1). This is ??.

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Bibliography