

ord-arithmetic.1 Ordinal Exponentiation

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We now move to ordinal exponentiation. Sadly, there is no *nice* synthetic definition for ordinal exponentiation.

Sure, there *are* explicit synthetic definitions. Here is one. Let $\text{finfun}(\alpha, \beta)$ be the set of all functions $f: \alpha \rightarrow \beta$ such that $\{\gamma \in \alpha : f(\gamma) \neq 0\}$ is equinumerous with some natural number. Define a well-ordering on $\text{finfun}(\alpha, \beta)$ by $f \sqsubset g$ iff $f \neq g$ and $f(\gamma_0) < g(\gamma_0)$, where $\gamma_0 = \max\{\gamma \in \alpha : f(\gamma) \neq g(\gamma)\}$. Then we can define $\alpha^{(\beta)}$ as $\text{ord}(\text{finfun}(\alpha, \beta), \sqsubset)$. Potter employs this explicit definition, and then immediately explains:

The choice of this ordering is determined purely by our desire to obtain a definition of ordinal exponentiation which obeys the appropriate recursive condition. . . , and it is much harder to picture than either the ordered sum or the ordered product. (Potter, 2004, p. 199)

Quite. We explained addition as “a copy of α followed by a copy of β ”, and multiplication as “a β -sequence of copies of α ”. But we have nothing pithy to say about $\text{finfun}(\alpha, \gamma)$. So instead, we’ll offer the definition of ordinal exponentiation just *by* transfinite recursion, i.e.:

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Definition ord-arithmetic.1.

$$\begin{aligned}\alpha^{(0)} &= 1 \\ \alpha^{(\beta+1)} &= \alpha^{(\beta)} \cdot \alpha \\ \alpha^{(\beta)} &= \bigcup_{\delta < \beta} \alpha^{(\delta)} \quad \text{when } \beta \text{ is a limit ordinal}\end{aligned}$$

If we were working *as* set theorists, we might want to explore some of the properties of ordinal exponentiation. But we have nothing much more to add, except to note the unsurprising fact that ordinal exponentiation does not commute. Thus $2^{(\omega)} = \bigcup_{\delta < \omega} 2^{(\delta)} = \omega$, whereas $\omega^{(2)} = \omega \cdot \omega$. But then, we should not *expect* exponentiation to commute, since it does not commute with natural numbers: $2^{(3)} = 8 < 9 = 3^{(2)}$.

Problem ord-arithmetic.1. Using Transfinite Induction, prove that, if we define $\alpha^{(\beta)} = \text{ord}(\text{finfun}(\alpha, \beta), \sqsubset)$, we obtain the recursion equations of **Definition ord-arithmetic.1**.

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Bibliography

Potter, Michael. 2004. *Set Theory and its Philosophy*. Oxford: Oxford University Press.