

## ord-arithmetic.1 Ordinal Addition

sth:ord-arithmetic:add:sec Suppose we want to add  $\alpha$  and  $\beta$ . We can simply put a copy of  $\beta$  immediately after a copy of  $\alpha$ . (We need to take *copies*, since we know from ?? that either  $\alpha \subseteq \beta$  or  $\beta \subseteq \alpha$ .) The intuitive effect of this is to run through an  $\alpha$ -sequence of steps, *and then* to run through a  $\beta$ -sequence. The resulting sequence will be well-ordered; so by ?? it is isomorphic to a (unique) ordinal. That ordinal can be regarded as the *sum* of  $\alpha$  and  $\beta$ .

That is the intuitive idea behind ordinal addition. To define it rigorously, we start with the idea of taking *copies* of sets. The idea here is to use arbitrary tags, 0 and 1, to keep track of which object came from where:

sth:ord-arithmetic:add: defdissum **Definition ord-arithmetic.1.** The *disjoint sum* of  $A$  and  $B$  is  $A \sqcup B = (A \times \{0\}) \cup (B \times \{1\})$ .

We next define an ordering on pairs of ordinals:

**Definition ord-arithmetic.2.** For any ordinals  $\alpha_1, \alpha_2, \beta_1, \beta_2$ , say that:

$$\langle \alpha_1, \alpha_2 \rangle \triangleleft \langle \beta_1, \beta_2 \rangle \text{ iff either } \alpha_2 \in \beta_2 \\ \text{or both } \alpha_2 = \beta_2 \text{ and } \alpha_1 \in \beta_1$$

This is a *reverse lexicographic* ordering, since you order by the second element, then by the first. Now recall that we wanted to define  $\alpha + \beta$  as the order type of a copy of  $\alpha$  followed by a copy of  $\beta$ . To achieve that, we say:

sth:ord-arithmetic:add: defordplus **Definition ord-arithmetic.3.** For any ordinals  $\alpha, \beta$ , their sum is  $\alpha + \beta = \text{ord}(\alpha \sqcup \beta, \triangleleft)$ .

Note that we slightly abused notation here; strictly we should write “ $\{\langle x, y \rangle \in \alpha \sqcup \beta : x \triangleleft y\}$ ” in place of “ $\triangleleft$ ”. For brevity, though, we will continue to abuse notation in this way in what follows.

The following result, together with ??, confirms that our definition is well-formed:

sth:ord-arithmetic:add: ordsumlessiswo **Lemma ord-arithmetic.4.**  $\langle \alpha \sqcup \beta, \triangleleft \rangle$  is a well-order, for any ordinals  $\alpha$  and  $\beta$ .

*Proof.* Obviously  $\triangleleft$  is connected on  $\alpha \sqcup \beta$ . To show it is well-founded, fix a non-empty  $X \subseteq \alpha \sqcup \beta$ . Let  $Y$  be the subset of  $X$  whose second coordinate is as small as possible, i.e.  $Y = \{\langle \gamma, i \rangle \in X : (\forall \langle \delta, j \rangle \in X) i \leq j\}$ . Now choose the element of  $Y$  with smallest first coordinate.  $\square$

So we have a nice, explicit definition of ordinal addition. Here is an unsurprising fact (recall that  $1 = \{0\}$ , by ??):

**Proposition ord-arithmetic.5.**  $\alpha + 1 = \alpha^+$ , for any ordinal  $\alpha$ .

*Proof.* Consider the isomorphism  $f$  from  $\alpha^+ = \alpha \cup \{\alpha\}$  to  $\alpha \sqcup 1 = (\alpha \times \{0\}) \sqcup (\{0\} \times \{1\})$  given by  $f(\gamma) = \langle \gamma, 0 \rangle$  for  $\gamma \in \alpha$ , and  $f(\alpha) = \langle 0, 1 \rangle$ .  $\square$

Moreover, it is easy to show that addition obeys certain recursive conditions:

**Lemma ord-arithmetic.6.** *For any ordinals  $\alpha, \beta$ , we have:*

[sth:ord-arithmetic:add:ordadditionrecursion](#)

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha + \beta &= \text{lsub}_{\delta < \beta}(\alpha + \delta) \quad \text{if } \beta \text{ is a limit ordinal} \end{aligned}$$

*Proof.* We check case-by-case; first:

$$\begin{aligned} \alpha + 0 &= \text{ord}((\alpha \times \{0\}) \cup (0 \times \{1\}), \triangleleft) \\ &= \text{ord}((\alpha \times \{0\}) \cup \{0\}, \triangleleft) \\ &= \alpha \\ \alpha + (\beta + 1) &= \text{ord}((\alpha \times \{0\}) \cup (\beta^+ \times \{1\}), \triangleleft) \\ &= \text{ord}((\alpha \times \{0\}) \cup (\beta \times \{1\}), \triangleleft) + 1 \\ &= (\alpha + \beta) + 1 \end{aligned}$$

Now let  $\beta \neq \emptyset$  be a limit. If  $\delta < \beta$  then also  $\delta + 1 < \beta$ , so  $\alpha + \delta$  is a proper initial segment of  $\alpha + \beta$ . So  $\alpha + \beta$  is a strict upper bound on  $X = \{\alpha + \delta : \delta < \beta\}$ . Moreover, if  $\alpha \leq \gamma < \alpha + \beta$ , then clearly  $\gamma = \alpha + \delta$  for some  $\delta < \beta$ . So  $\alpha + \beta = \text{lsub}_{\delta < \beta}(\alpha + \delta)$ .  $\square$

But here is a striking fact. To define ordinal addition, we could *instead* have simply used the Transfinite Recursion Theorem, and laid down the recursion equations, exactly as given in **Lemma ord-arithmetic.6** (though using “ $\beta^+$ ” rather than “ $\beta + 1$ ”).

There are, then, two different ways to define operations on the ordinals. We can define them *synthetically*, by explicitly constructing a well-ordered set and considering its order type. Or we can define them *recursively*, just by laying down the recursion equations. Done correctly, though, the outcome is identical. For ?? guarantees that these recursion equations pin down *unique* ordinals.

In many ways, ordinal arithmetic behaves just like addition of the natural numbers. For example, we can prove the following:

**Lemma ord-arithmetic.7.** *If  $\alpha, \beta, \gamma$  are ordinals, then:*

[sth:ord-arithmetic:add:ordinaladditionisnice](#)

1. *if  $\beta < \gamma$ , then  $\alpha + \beta < \alpha + \gamma$*

[sth:ord-arithmetic:add:ordaddition1](#)

2. *if  $\alpha + \beta = \alpha + \gamma$ , then  $\beta = \gamma$*

[sth:ord-arithmetic:add:ordaddition2](#)

3.  *$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ , i.e., addition is associative*

[sth:ord-arithmetic:add:ordaddition3](#)

4. *If  $\alpha \leq \beta$ , then  $\alpha + \gamma \leq \beta + \gamma$*

[sth:ord-arithmetic:add:ordaddition4](#)

*Proof.* We prove (3), leaving the rest as an exercise. The proof is by Simple Transfinite Induction on  $\gamma$ , using Lemma ord-arithmetic.6. When  $\gamma = 0$ :

$$(\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0)$$

When  $\gamma = \delta + 1$ , suppose for induction that  $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$ ; now using Lemma ord-arithmetic.6 three times:

$$\begin{aligned} (\alpha + \beta) + (\delta + 1) &= ((\alpha + \beta) + \delta) + 1 \\ &= (\alpha + (\beta + \delta)) + 1 \\ &= \alpha + ((\beta + \delta) + 1) \\ &= \alpha + (\beta + (\delta + 1)) \end{aligned}$$

When  $\gamma$  is a limit ordinal, suppose for induction that if  $\delta \in \gamma$  then  $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$ ; now:

$$\begin{aligned} (\alpha + \beta) + \gamma &= \text{lsub}_{\delta < \gamma}((\alpha + \beta) + \delta) \\ &= \text{lsub}_{\delta < \gamma}(\alpha + (\beta + \delta)) \\ &= \alpha + \text{lsub}_{\delta < \gamma}(\beta + \delta) \\ &= \alpha + (\beta + \gamma) \end{aligned} \quad \square$$

**Problem ord-arithmetic.1.** Prove the remainder of Lemma ord-arithmetic.7.

In these ways, ordinal addition should be very familiar. But, there is a crucial way in which ordinal addition is *not* like addition on the natural numbers.

sth:ord-arithmetic:add:  
ordsumnotcommute **Proposition ord-arithmetic.8.** *Ordinal addition is not commutative;  $1 + \omega = \omega < \omega + 1$ .*

*Proof.* Note that  $1 + \omega = \text{lsub}_{n < \omega}(1 + n) = \omega \in \omega \cup \{\omega\} = \omega^+ = \omega + 1$ .  $\square$

Whilst this may initially come as a surprise, *it shouldn't*. On the one hand, when you consider  $1 + \omega$ , you are thinking about the order type you get by putting an extra element *before* all the natural numbers. Reasoning as we did with Hilbert's Hotel in ??, intuitively, this extra first element shouldn't make any difference to the overall order type. On the other hand, when you consider  $\omega + 1$ , you are thinking about the order type you get by putting an extra element *after* all the natural numbers. And that's a radically different beast!

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## Bibliography