

choice.1 Vitali's Paradox

sth:choice:vitali:
sec To get a real sense of whether the Banach-Tarski construction is acceptable or not, we should examine its *proof*. Unfortunately, that would require much more algebra than we can present here. However, we can offer some quick remarks which might shed some insight on the proof of Banach-Tarski,¹ by focussing on the following result:

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vitaliparadox **Theorem choice.1 (Vitali's Paradox (in ZFC)).** *Any circle can be decomposed into countably many pieces, which can be reassembled (by rotation and transportation) to form two copies of that circle.*

This is much easier to prove than the Banach–Tarski Paradox. We have called it “Vitali’s Paradox”, since it follows from Vitali’s 1905 construction of an unmeasurable set. But the set-theoretic aspects of the proof of Vitali’s Paradox and the Banach-Tarski Paradox are very similar. The essential difference between the results is just that Banach-Tarski considers a *finite* decomposition, whereas Vitali’s Paradox considers a *countably infinite* decomposition. As Weston (2003) puts it, Vitali’s Paradox “is certainly not nearly as striking as the Banach–Tarski paradox, but it does illustrate that geometric paradoxes can happen even in ‘simple’ situations.”

Vitali’s Paradox concerns a two-dimensional figure, a circle. So we will work on the plane, \mathbb{R}^2 . Let R be the set of (clockwise) rotations of points around the origin by *rational* radian values between $[0, 2\pi)$. Here are some algebraic facts about R (if you don’t understand the statement of the result, the proof will make its meaning clear):

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rotationsgroupabelian **Lemma choice.2.** *R forms an abelian group under composition of functions.*

Proof. Writing 0_R for the rotation by 0 radians, this is an identity element for R , since $\rho \circ 0_R = 0_R \circ \rho = \rho$ for any $\rho \in R$.

Every element has an inverse. Where $\rho \in R$ rotates by r radians, $\rho^{-1} \in R$ rotates by $2\pi - r$ radians, so that $\rho \circ \rho^{-1} = 0_R$.

Composition is associative: $(\tau \circ \sigma) \circ \rho = \tau \circ (\sigma \circ \rho)$ for any $\rho, \sigma, \tau \in R$

Composition is commutative: $\sigma \circ \rho = \rho \circ \sigma$ for any $\rho, \sigma \in R$. □

In fact, we can split our group R in half, and then use either half to recover the whole group:

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disjointgroup **Lemma choice.3.** *There is a partition of R into two disjoint sets, R_1 and R_2 , both of which are a basis for R .*

Proof. Let R_1 consist of the rotations by rational radian values in $[0, \pi)$; let $R_2 = R \setminus R_1$. By elementary algebra, $\{\rho \circ \rho : \rho \in R_1\} = R$. A similar result can be obtained for R_2 . □

¹For a much fuller treatment, see Weston (2003) or Tomkowicz and Wagon (2016).

We will use this fact about groups to establish [Theorem choice.1](#). Let \mathbf{S} be the unit circle, i.e., the set of points exactly 1 unit away from the origin of the plane, i.e., $\{(r, s) \in \mathbb{R}^2 : \sqrt{r^2 + s^2} = 1\}$. We will split \mathbf{S} into parts by considering the following relation on \mathbf{S} :

$$r \sim s \text{ iff } (\exists \rho \in R)\rho(r) = s.$$

That is, the points of \mathbf{S} are linked by this relation iff you can get from one to the other by a rational-valued rotation about the origin. Unsurprisingly:

Lemma choice.4. \sim is an equivalence relation.

Proof. Trivial, using [Lemma choice.2](#). □

We now invoke Choice to obtain a set, C , containing exactly one member from each equivalence class of \mathbf{S} under \sim . That is, we consider a choice function f on the set of equivalence classes,²

$$E = \{[r]_{\sim} : r \in \mathbf{S}\},$$

and let $C = \text{ran}(f)$. For each rotation $\rho \in R$, the set $\rho[C]$ consists of the points obtained by applying the rotation ρ to each point in C . These next two results show that these sets cover the circle completely and without overlap:

Lemma choice.5. $\mathbf{S} = \bigcup_{\rho \in R} \rho[C]$.

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Proof. Fix $s \in \mathbf{S}$; there is some $r \in C$ such that $r \in [s]_{\sim}$, i.e., $r \sim s$, i.e., $\rho(r) = s$ for some $\rho \in R$. □

Lemma choice.6. If $\rho_1 \neq \rho_2$ then $\rho_1[C] \cap \rho_2[C] = \emptyset$.

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Proof. Suppose $s \in \rho_1[C] \cap \rho_2[C]$. So $s = \rho_1(r_1) = \rho_2(r_2)$ for some $r_1, r_2 \in C$. Hence $\rho_2^{-1}(\rho_1(r_1)) = r_2$, and $\rho_2^{-1} \circ \rho_1 \in R$, so $r_1 \sim r_2$. So $r_1 = r_2$, as C selects exactly one member from each equivalence class under \sim . So $s = \rho_1(r_1) = \rho_2(r_1)$, and hence $\rho_1 = \rho_2$. □

We now apply our earlier algebraic facts to our circle:

Lemma choice.7. There is a partition of \mathbf{S} into two disjoint sets, D_1 and D_2 , such that D_1 can be partitioned into countably many sets which can be rotated to form a copy of \mathbf{S} (and similarly for D_2).

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²Note: since R is **enumerable**, each **element** of E is **enumerable**. Since \mathbf{S} is **non-enumerable**, it follows from [Lemma choice.5](#) and ?? that E is **non-enumerable**. So this is a use of *uncountable* Choice.

Proof. Using R_1 and R_2 from [Lemma choice.3](#), let:

$$D_1 = \bigcup_{\rho \in R_1} \rho[C] \qquad D_2 = \bigcup_{\rho \in R_1} \rho[C]$$

This is a partition of \mathbf{S} , by [Lemma choice.5](#), and D_1 and D_2 are disjoint by [Lemma choice.6](#). By construction, D_1 can be partitioned into countably many sets, $\rho[C]$ for each $\rho \in R_1$. And these can be rotated to form a copy of \mathbf{S} , since $\mathbf{S} = \bigcup_{\rho \in R} \rho[C] = \bigcup_{\rho \in R_1} (\rho \circ \rho)[C]$ by [Lemma choice.3](#) and [Lemma choice.5](#). The same reasoning applies to D_2 . \square

This immediately entails Vitali’s Paradox. For we can generate *two* copies of \mathbf{S} from \mathbf{S} , just by splitting it up into countably many pieces (the various $\rho[C]$ ’s) and then rigidly moving them (simply rotate each piece of D_1 , and first transport and then rotate each piece of D_2).

Let’s recap the proof-strategy. We started with some algebraic facts about the group of rotations on the plane. We used this group to partition \mathbf{S} into equivalence classes. We then arrived at a “paradox”, by using Choice to select elements from each class.

We use exactly the same strategy to prove Banach–Tarski. The main difference is that the algebraic facts used to prove Banach–Tarski are significantly more complicated than those used to prove Vitali’s Paradox. But those algebraic facts have nothing to do with Choice. We will summarise them quickly.

To prove Banach–Tarski, we start by establishing an analogue of [Lemma choice.3](#): any *free group* can be split into four pieces, which intuitively we can “move around” to recover two copies of the whole group.³ We then show that we can use two particular rotations around the origin of \mathbb{R}^3 to generate a free group of rotations, F .⁴ (No Choice yet.) We now regard points on the surface of the sphere as “similar” iff one can be obtained from the other by a rotation in F . We then *use Choice* to select exactly one point from each equivalence class of “similar” points. Applying our division of F to the surface of the sphere, as in [Lemma choice.7](#), we split that surface into four pieces, which we can “move around” to obtain two copies of the surface of the sphere. And this establishes ([Hausdorff, 1914](#)):

Theorem choice.8 (Hausdorff’s Paradox (in ZFC)). *The surface of any sphere can be decomposed into finitely many pieces, which can be reassembled (by rotation and transportation) to form two disjoint copies of that sphere.*

A couple of further algebraic tricks are needed to obtain the full Banach–Tarski Theorem (which concerns not just the sphere’s surface, but its interior too). Frankly, however, this is just icing on the algebraic cake. Hence Weston writes:

³The fact that we can use *four* pieces is due to [Robinson \(1947\)](#). For a recent proof, see [Tomkowicz and Wagon \(2016, Theorem 5.2\)](#). We follow [Weston \(2003, p. 3\)](#) in describing this as “moving” the pieces of the group.

⁴See [Tomkowicz and Wagon \(2016, Theorem 2.1\)](#).

[...] the result on free groups is the *key step* in the proof of the Banach-Tarski paradox. From this point of view, the Banach-Tarski paradox is not a statement about \mathbb{R}^3 so much as it is a statement about the complexity of the group [of translations and rotations in \mathbb{R}^3]. (Weston, 2003, p. 16)

That is: whether we can offer a *finite* decomposition (as in Banach-Tarski) or a *countably infinite* decomposition (as in Vitali's Paradox) comes down to certain group-theoretic facts about working in two-dimension or three-dimensions.

Admittedly, this last observation slightly spoils the joke at the end of ???. Since it is two dimensional, "Banach-Tarski" must be divided into a countable *infinity* of pieces, if one wants to rearrange those pieces to form "Banach-Tarski Banach-Tarski". To repair the joke, one must write in three dimensions. We leave this as an exercise for the reader.

One final comment. In ???, we mentioned that the "pieces" of the sphere one obtains cannot be *measurable*, but must be unpicturable "infinite scatterings". The same is true of our use of Choice in obtaining [Lemma choice.7](#). And this is all worth explaining.

Again, we must sketch some background (but this is *just* a sketch; you may want to consult a textbook entry on *measure*). To define a measure for a set X is to assign a value $\mu(E) \in \mathbb{R}$ for each E in some " σ -algebra" on X . Details here are not essential, except that the function μ must obey the principle of countable additivity: the measure of a countable union of disjoint sets is the sum of their individual measures, i.e., $\mu(\bigcup_{n < \omega} X_n) = \sum_{n < \omega} \mu(X_n)$ whenever the X_n s are disjoint. To say that a set is "unmeasurable" is to say that no measure can be suitably assigned. Now, using our R from before:

Corollary choice.9 (Vitali). *Let μ be a measure such that $\mu(\mathbf{S}) = 1$, and such that $\mu(X) = \mu(Y)$ if X and Y are congruent. Then $\rho[C]$ is unmeasurable for all $\rho \in R$.*

Proof. For reductio, suppose otherwise. So let $\mu(\sigma[C]) = r$ for some $\sigma \in R$ and some $r \in \mathbb{R}$. For any $\rho \in C$, $\rho[C]$ and $\sigma[C]$ are congruent, and hence $\mu(\rho[C]) = r$ for any $\rho \in C$. By [Lemma choice.5](#) and [Lemma choice.6](#), $\mathbf{S} = \bigcup_{\rho \in R} \rho[C]$ is a countable union of pairwise disjoint sets. So countable additivity dictates that $\mu(\mathbf{S}) = 1$ is the sum of the measures of each $\rho[C]$, i.e.,

$$1 = \mu(\mathbf{S}) = \sum_{\rho \in R} \mu(\rho[C]) = \sum_{\rho \in R} r$$

But if $r = 0$ then $\sum_{\rho \in R} r = 0$, and if $r > 0$ then $\sum_{\rho \in R} r = \infty$. □

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