choice.1 The Tarski-Scott Trick

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In ??, we defined cardinals as ordinals. To do this, we assumed the Axiom of Well-Ordering. We did this, for no other reason than that it is the "industry standard".

Before we discuss any of the philosophical issues surrounding Well-Ordering, then, it is important to be clear that we *can* depart from the industry standard, and develop a theory of cardinals *without* assuming Well-Ordering. We can still employ the definitions of $A \approx B$, $A \preceq B$ and $A \prec B$, as they appeared in ??. We will just need a new notion of *cardinal*.

A naïve thought would be to attempt to define A's cardinality thus:

$$\{x : A \approx x\}.$$

You might want to compare this with Frege's definition of $\#xFx$, sketched at the very end of ??. And, for reasons we gestured at there, this definition fails. Any singleton set is equinumerous with $\{\emptyset\}$. But new singleton sets are formed at every successor stage of the hierarchy (just consider the singleton of the previous stage). So $\{x : A \approx x\}$ does not exist, since it cannot have a rank.

To get around this problem, we use a trick due to Tarski and Scott:

Definition choice.1 (Tarski-Scott). For any formula $\varphi(x)$,[1] let $[x : \varphi(x)]$ be the set of all x, of least possible rank, such that $\varphi(x)$ (or $\emptyset$, if there are no φs).

We should check that this definition is legitimate. Working in **ZF**, ?? guarantees that $\text{rank}(x)$ exists for every x. Now, if there are any entities satisfying φ, then we can let α be the least rank such that $(\exists x \subseteq V_\alpha)\varphi(x)$, i.e., $(\forall \beta \in \alpha)(\forall x \subseteq V_\beta)\neg\varphi(x)$. We can then define $[x : \varphi(x)]$ by Separation as $\{x \in V_{\alpha+1} : \varphi(x)\}$.

Having justified the Tarski-Scott trick, we can now use it to define a notion of cardinality:

Definition choice.2. The TS-cardinality of A is $\text{tsc}(A) = [x : A \approx x]$.

The definition of a TS-cardinal does not use Well-Ordering. But, even without that Axiom, we can show that TS-*cardinals* behave rather like *cardinals* as defined in ??. For example, if we restate ?? and ?? in terms of TS-cardinals, the proofs go through just fine in **ZF**, without assuming Well-Ordering.

Whilst we are on the topic, it is worth noting that we can also develop a theory of ordinals using the Tarski-Scott trick. Where $\langle A, <\rangle$ is a well-ordering, let $\text{tso}(A, <) = [\langle X, R\rangle : \langle A, <\rangle \cong \langle X, R\rangle]$. For more on this treatment of cardinals and ordinals, see Potter (2004, chs. 9–12).

[1] Which may have parameters.

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Bibliography

Potter, Michael. 2004. *Set Theory and its Philosophy*. Oxford: Oxford University Press.