choice.1 Comparability and Hartogs’ Lemma

That’s the plus side. Here’s the minus side. Without Choice, things get messy. To see why, here is a nice result due to Hartogs (1915):

\[ \text{Lemma choice.1 (in ZF). For any set } A, \text{ there is an ordinal } \alpha \text{ such that } \alpha \not\in A \]

\[ \text{Proof. If } B \subseteq A \text{ and } R \subseteq B^2, \text{ then } \langle B, R \rangle \subseteq V_{\text{rank}(A)+4} \text{ by } \text{??}. \text{ So, using Separation, consider:} \]

\[ C = \{ \langle B, R \rangle \in V_{\text{rank}(A)+5} : B \subseteq A \text{ and } \langle B, R \rangle \text{ is a well-ordering} \} \]

Using Replacement and ??, form the set:

\[ \alpha = \{ \text{ord}(B, R) : \langle B, R \rangle \in C \}. \]

By ??, \( \alpha \) is an ordinal, since it is a transitive set of ordinals. After all, if \( \gamma \in \beta \in \alpha \), then \( \beta = \text{ord}(B, R) \) for some \( B \subseteq R \), whereupon \( \gamma = \text{ord}(B_b, R_b) \) for some \( b \in B \) by ??, so that \( \gamma \in \alpha \).

For reductio, suppose there is an injection \( f : \alpha \to A \). Then, where:

\[ B = \text{ran}(f) \]
\[ R = \{ (f(\alpha), f(\beta)) \in A \times A : \alpha \in \beta \}. \]

Clearly \( \alpha = \text{ord}(B, R) \) and \( \langle B, R \rangle \in C \). So \( \alpha \in \alpha \), which is a contradiction. \( \square \)

This entails a deep result:

\[ \text{Theorem choice.2 (in ZF). The following claims are equivalent:} \]

1. The Axiom of Well-Ordering

\[ \text{Proof. (1) } \Rightarrow (2). \text{ Fix } A \text{ and } B. \text{ Invoking (1), there are well-orderings } \langle A, R \rangle \text{ and } \langle B, S \rangle. \text{ Invoking } ???, \text{ let } f : \alpha \to \langle A, R \rangle \text{ and } g : \beta \to \langle B, S \rangle \text{ be isomorphisms. By } ???, \text{ either } \alpha \subseteq \beta \text{ or } \beta \subseteq \alpha. \text{ If } \alpha \subseteq \beta, \text{ then } g \circ f^{-1} : A \to B \text{ is an injection, and hence } A \preceq B; \text{ similarly, if } \beta \subseteq \alpha \text{ then } B \preceq A. \]

(2) \( \Rightarrow (1). \text{ Fix } A; \text{ by Lemma choice.1 there is some ordinal } \beta \text{ such that } \beta \not\in A. \text{ Invoking (2), we have } A \preceq \beta. \text{ So there is some injection } f : A \to \beta, \text{ and we can use this injection to well-order the elements of } A, \text{ by defining an order } \{ (a, b) \in A \times A : f(a) \in f(b) \}. \] \( \square \)

As an immediate consequence: if Well-Ordering fails, then some sets are literally incomparable with regard to their size. So, if Well-Ordering fails, then transfinite cardinal arithmetic will be messy. For example, we will have to abandon the idea that if \( A \) and \( B \) are infinite then \( A \cup B \approx A \times B \approx M \), where
M is the larger of A and B (see ??). The problem is simple: if we cannot compare the size of A and B, then it is nonsensical to ask which is larger.

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Bibliography