

## cardinals.1 Finite, Enumerable, Non-enumerable

sth:cardinals:classification:sec

Now that we have been introduced to cardinals, it is worth spending a little time talking about different varieties of cardinals; specifically, finite, **enumerable**, and **non-enumerable** cardinals.

Our first two results entail that the finite cardinals will be exactly the finite ordinals, which we defined as our *natural numbers* back in ??:

sth:cardinals:classification:finitecardisoequal

**Proposition cardinals.1.** *Let  $n, m \in \omega$ . Then  $n = m$  iff  $n \approx m$ .*

*Proof.* *Left-to-right* is trivial. To prove *right-to-left*, suppose  $n \approx m$  although  $n \neq m$ . By Trichotomy, either  $n \in m$  or  $m \in n$ ; suppose  $n \in m$  without loss of generality. Then  $n \subsetneq m$  and there is a **bijection**  $f: m \rightarrow n$ , so that  $m$  is Dedekind infinite, contradicting ??.  $\square$

sth:cardinals:classification:naturalsarecardinals

**Corollary cardinals.2.** *If  $n \in \omega$ , then  $n$  is a cardinal.*

*Proof.* Immediate.  $\square$

It also follows that several reasonable notions of what it might mean to describe a cardinal as “finite” or “infinite” coincide:

sth:cardinals:classification:generalinfinitycharacter

**Theorem cardinals.3.** *For any set  $A$ , the following are equivalent:*

sth:cardinals:classification:card:notinomega

1.  $|A| \notin \omega$ , i.e.,  $A$  is not a natural number;

sth:cardinals:classification:card:omegaplus

2.  $\omega \leq |A|$ ;

sth:cardinals:classification:card:infinite

3.  $A$  is Dedekind infinite.

*Proof.* From ??, ??, and **Corollary cardinals.2**.  $\square$

This licenses the following *definition* of some notions which we used rather informally in ??:

sth:cardinals:classification:definite

**Definition cardinals.4.** We say that  $A$  is *finite* iff  $|A|$  is a natural number, i.e.,  $|A| \in \omega$ . Otherwise, we say that  $A$  is *infinite*.

But note that this definition is presented against the background of **ZFC**. After all, we needed Well-Ordering to guarantee that every set has a cardinality. And indeed, without Well-Ordering, there can be a set which is neither finite nor Dedekind infinite. We will return to this sort of issue in ??. For now, we continue to rely upon Well-Ordering.

Let us now turn from the finite cardinals to the infinite cardinals. Here are two elementary points:

sth:cardinals:classification:omegaisacardinal

**Corollary cardinals.5.**  $\omega$  is the least infinite cardinal.

*Proof.*  $\omega$  is a cardinal, since  $\omega$  is Dedekind infinite and if  $\omega \approx n$  for any  $n \in \omega$  then  $n$  would be Dedekind infinite, contradicting ??. Now  $\omega$  is the least infinite cardinal by definition.  $\square$

**Corollary cardinals.6.** *Every infinite cardinal is a limit ordinal.*

*Proof.* Let  $\alpha$  be an infinite successor ordinal, so  $\alpha = \beta + 1$  for some  $\beta$ . By **Proposition cardinals.1**,  $\beta$  is also infinite, so  $\beta \approx \beta + 1$  by ???. Now  $|\beta| = |\beta + 1| = |\alpha|$  by ???, so that  $\alpha \neq |\alpha|$ .  $\square$

Now, as early as ???, we flagged we can distinguish between **enumerable** and **non-enumerable** infinite sets. That definition naturally leads to the following:

**Proposition cardinals.7.** *A is enumerable iff  $|A| \leq \omega$ , and A is non-enumerable iff  $\omega < |A|$ .*

*Proof.* By Trichotomy, the two claims are equivalent, so it suffices to prove that  $A$  is **enumerable** iff  $|A| \leq \omega$ . For *right-to-left*: if  $|A| \leq \omega$ , then  $A \preceq \omega$  by ??? and **Corollary cardinals.5**. For *left-to-right*: suppose  $A$  is **enumerable**; then by ??? there are three possible cases:

1. if  $A = \emptyset$ , then  $|A| = 0 \in \omega$ , by **Corollary cardinals.2** and ???.
2. if  $n \approx A$ , then  $|A| = n \in \omega$ , by **Corollary cardinals.2** and ???.
3. if  $\omega \approx A$ , then  $|A| = \omega$ , by **Corollary cardinals.5**.

So in all cases,  $|A| \leq \omega$ .  $\square$

Indeed,  $\omega$  has a special place. Whilst there are many countable ordinals:

**Corollary cardinals.8.**  *$\omega$  is the only enumerable infinite cardinal.*

*Proof.* Let  $\mathfrak{a}$  be a **enumerable** infinite cardinal. Since  $\mathfrak{a}$  is infinite,  $\omega \leq \mathfrak{a}$ . Since  $\mathfrak{a}$  is a **enumerable** cardinal,  $\mathfrak{a} = |\mathfrak{a}| \leq \omega$ . So  $\mathfrak{a} = \omega$  by Trichotomy.  $\square$

Evidently there are infinitely many cardinals. So we might ask: *How many cardinals are there?* The following results show that we might want to reconsider that question.

**Proposition cardinals.9.** *If every member of  $X$  is a cardinal, then  $\bigcup X$  is a cardinal.* *sth:cardinals:classification: unioncardinalscardinal*

*Proof.* It is easy to check that  $\bigcup X$  is an ordinal. Let  $\alpha \in \bigcup X$  be an ordinal; then  $\alpha \in \mathfrak{b} \in X$  for some cardinal  $\mathfrak{b}$ . Since  $\mathfrak{b}$  is a cardinal,  $\alpha < \mathfrak{b}$ . Since  $\mathfrak{b} \subseteq \bigcup X$ , we have  $\mathfrak{b} \preceq \bigcup X$ , and so  $\alpha \not\approx \bigcup X$ . Generalising,  $\bigcup X$  is a cardinal.  $\square$

**Theorem cardinals.10.** *There is no largest cardinal.* *sth:cardinals:classification: lem:NoLargestCardinal*

*Proof.* For any cardinal  $\mathfrak{a}$ , Cantor's Theorem (??) and ??? entail that  $\mathfrak{a} < |\wp(\mathfrak{a})|$ .  $\square$

**Theorem cardinals.11.** *The set of all cardinals does not exist.*

*Proof.* For reductio, suppose  $C = \{\mathfrak{a} : \mathfrak{a} \text{ is a cardinal}\}$ . Now  $\bigcup C$  is a cardinal by [Proposition cardinals.9](#), so by [Theorem cardinals.10](#) there is a cardinal  $\mathfrak{b} > \bigcup C$ . By definition  $\mathfrak{b} \in C$ , so  $\mathfrak{b} \subseteq \bigcup C$ , so that  $\mathfrak{b} \leq \bigcup C$ , a contradiction.  $\square$

You should compare this with both Russell's Paradox and Burali-Forti.

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## Bibliography