

cardinals.1 Cardinals as Ordinals

sth:cardinals:cardsasords:
sec

In fact, our theory of cardinals will just make (shameless) use of our theory of ordinals. That is: we will just define cardinals as certain specific ordinals. In particular, we will offer the following:

sth:cardinals:cardsasords:
defcardinalasordinal

Definition cardinals.1. If A can be well-ordered, then $|A|$ is the least ordinal γ such that $A \approx \gamma$. For any ordinal γ , we say that γ is a *cardinal* iff $\gamma = |\gamma|$.

We just used the phrase “ A can be well-ordered”. As is almost always the case in mathematics, the modal locution here is just a hand-waving gloss on an existential claim: to say “ A can be well-ordered” is just to say “there is a relation which well-orders A ”.

But there is a snag with **Definition cardinals.1**. We would like it to be the case that *every* set has a size, i.e., that $|A|$ exists for every A . The definition we just gave, though, begins with a conditional: “*If* A can be well-ordered. . .”. If there is some set A which cannot be well-ordered, then our definition will simply fail to define an object $|A|$.

So, to use **Definition cardinals.1**, we need a guarantee that every set can be well-ordered. Sadly, though, this guarantee is unavailable in **ZF**. So, if we want to use **Definition cardinals.1**, there is no alternative but to add a new axiom, such as:

Axiom (Well-Ordering). Every set can be well-ordered.

We will discuss whether the Well-Ordering Axiom is acceptable in **??**. From now on, though, we will simply help ourselves to it. And, using it, it is quite straightforward to prove that cardinals (as defined in **Definition cardinals.1**) exist and behave nicely:

sth:cardinals:cardsasords:
lem:CardinalsExist

Lemma cardinals.2. For every set A :

sth:cardinals:cardsasords:
cardaexists

1. $|A|$ exists and is unique;

sth:cardinals:cardsasords:
cardaapprox

2. $|A| \approx A$;

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cardaidem

3. $|A|$ is a cardinal, i.e., $|A| = ||A||$;

Proof. Fix A . By Well-Ordering, there is a well-ordering $\langle A, R \rangle$. By **??**, $\langle A, R \rangle$ is isomorphic to a unique ordinal, β . So $A \approx \beta$. By Transfinite Induction, there is a uniquely least ordinal, γ , such that $A \approx \gamma$. So $|A| = \gamma$, establishing (1) and (2). To establish (3), note that if $\delta \in \gamma$ then $\delta \prec A$, by our choice of γ , so that also $\delta \prec \gamma$ since equinumerosity is an equivalence relation (**??**). So $\gamma = |\gamma|$. \square

The next result guarantees Cantor’s Principle, and more besides. (Note that cardinals inherit their ordering from the ordinals, i.e., $\mathfrak{a} < \mathfrak{b}$ iff $\mathfrak{a} \in \mathfrak{b}$. In formulating this, we will use Fraktur letters for objects we know to be cardinals. This is fairly standard. A common alternative is to use Greek letters, since

cardinals are ordinals, but to choose them from the middle of the alphabet, e.g.: κ, λ):

Lemma cardinals.3. *For any sets A and B :*

*sth:cardinals:cardinalsords:
lem:CardinalsBehaveRight*

$$A \approx B \text{ iff } |A| = |B|$$

$$A \preceq B \text{ iff } |A| \leq |B|$$

$$A \prec B \text{ iff } |A| < |B|$$

Proof. We will prove the left-to-right direction of the second claim (the other cases are similar, and left as an exercise). So, consider the following diagram:



The double-headed arrows indicate **bijections**, whose existence is guaranteed by **Lemma cardinals.2**. In assuming that $A \preceq B$, there is some **an injection** to $A \rightarrow B$. Now, chasing the arrows around from $|A|$ to A to B to $|B|$, we obtain **an injection** $|A| \rightarrow |B|$ (the dashed arrow). \square

We can also use **Lemma cardinals.3** to re-prove Schröder–Bernstein. This is the claim that if $A \preceq B$ and $B \preceq A$ then $A \approx B$. We stated this as **??**, but first proved it—with some effort—in **??**. Now consider:

Re-proof. If $A \preceq B$ and $B \preceq A$, then $|A| \leq |B|$ and $|B| \leq |A|$ by **Lemma cardinals.3**. So $|A| = |B|$ and $A \approx B$ by Trichotomy and **Lemma cardinals.3**. \square

Whilst this is a very simple proof, it implicitly relies on both Replacement (to secure **??**) and on Well-Ordering (to guarantee **Lemma cardinals.3**). By contrast, the proof of **??** was much more self-standing (indeed, it can be carried out in \mathbf{Z}^-).

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Bibliography