cardinals.1 Cardinals as Ordinals

sth:cardinals:cardsasords:sec

In fact, our theory of cardinals will just make (shameless) use of our theory of ordinals. That is: we will just define cardinals as certain specific ordinals. In particular, we will offer the following:

sth:cardinals:cardsasords:defcardinalasordinal

Definition cardinals.1. If A can be well-ordered, then $|A|$ is the least ordinal γ such that $A \approx \gamma$. For any ordinal γ, we say that γ is a *cardinal* iff $\gamma = |\gamma|$.

We just used the phrase "A can be well-ordered". As is almost always the case in mathematics, the modal locution here is just a hand-waving gloss on an existential claim: to say "A can be well-ordered" is just to say "there is a relation which well-orders A".

But there is a snag with Definition cardinals.1. We would like it to be the case that *every* set has a size, i.e., that $|A|$ exists for every A. The definition we just gave, though, begins with a conditional: "*If* A can be well-ordered…". If there is some set A which cannot be well-ordered, then our definition will simply fail to define an object $|A|$.

So, to use Definition cardinals.1, we need a guarantee that every set can be well-ordered. Sadly, though, this guarantee is unavailable in **ZF**. So, if we want to use Definition cardinals.1, there is no alternative but to add a new axiom, such as:

Axiom (Well-Ordering). Every set can be well-ordered.

We will discuss whether the Well-Ordering Axiom is acceptable in **??**. From now on, though, we will simply help ourselves to it. And, using it, it is quite straightforward to prove that cardinals (as defined in Definition cardinals.1) exist and behave nicely:

sth:cardinals:cardsasords:lem:CardinalsExist

Lemma cardinals.2. *For every set A:*

sth:cardinals:cardsasords:cardaexists

1. *$|A|$ exists and is unique;*

sth:cardinals:cardsasords:cardaapprox

2. *$|A| \approx A$;*

sth:cardinals:cardsasords:cardaidem

3. *$|A|$ is a cardinal, i.e., $|A| = ||A||$;*

Proof. Fix A. By Well-Ordering, there is a well-ordering $\langle A, R\rangle$. By **??**, $\langle A, R\rangle$ is isomorphic to a unique ordinal, β. So $A \approx \beta$. By Transfinite Induction, there is a uniquely least ordinal, γ, such that $A \approx \gamma$. So $|A| = \gamma$, establishing (1) and (2). To establish (3), note that if $\delta \in \gamma$ then $\delta \prec A$, by our choice of γ, so that also $\delta \prec \gamma$ since equinumerosity is an equivalence relation (**??**). So $\gamma = |\gamma|$. □

The next result guarantees Cantor's Principle, and more besides. (Note that cardinals inherit their ordering from the ordinals, i.e., $\mathfrak{a} < \mathfrak{b}$ iff $\mathfrak{a} \in \mathfrak{b}$. In formulating this, we will use Fraktur letters for objects we know to be cardinals. This is fairly standard. A common alternative is to use Greek letters, since cardinals are ordinals, but to choose them from the middle of the alphabet, e.g.: κ, λ.):

Lemma cardinals.3. *For any sets A and B:* sth:cardinals:cardsasords:lem:CardinalsBehaveRight

$$\begin{aligned} A \approx B &\text{ iff } |A| = |B| \\ A \preceq B &\text{ iff } |A| \leq |B| \\ A \prec B &\text{ iff } |A| < |B| \end{aligned}$$

Proof. We will prove the left-to-right direction of the second claim (the other cases are similar, and left as an exercise). So, consider the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, leftrightarrow] & B \arrow[d, leftrightarrow] \\ {|A|} \arrow[r, dashed] & {|B|} \end{tikzcd}$$

The double-headed arrows indicate bijections, whose existence is guaranteed by Lemma cardinals.2. In assuming that $A \preceq B$, there is an injection $A \to B$. Now, chasing the arrows around from $|A|$ to A to B to $|B|$, we obtain an injection $|A| \to |B|$ (the dashed arrow). □

We can also use Lemma cardinals.3 to re-prove Schröder–Bernstein. This is the claim that if $A \preceq B$ and $B \preceq A$ then $A \approx B$. We stated this as **??**, but first proved it—with some effort—in **??**. Now consider:

Re-proof of Schröder-Bernstein. If $A \preceq B$ and $B \preceq A$, then $|A| \leq |B|$ and $|B| \leq |A|$ by Lemma cardinals.3. So $|A| = |B|$ and $A \approx B$ by Trichotomy and Lemma cardinals.3. □

Whilst this is a very simple proof, it implicitly relies on both Replacement (to secure **??**) and on Well-Ordering (to guarantee Lemma cardinals.3). By contrast, the proof of **??** was much more self-standing (indeed, it can be carried out in $\mathbf{Z}^-$).

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Bibliography