It turns out that transfinite cardinal addition and multiplication is extremely easy. This follows from the fact that cardinals are (certain) ordinals, and so well-ordered, and so can be manipulated in a certain way. Showing this, though, is not so easy. To start, we need a tricksy definition:

**Definition card-arithmetic.1.** We define a canonical ordering, \(<\), on pairs of ordinals, by stipulating that \(\langle \alpha_1, \alpha_2 \rangle \preccurlyeq \langle \beta_1, \beta_2 \rangle\) iff either:

1. \(\max(\alpha_1, \alpha_2) < \max(\beta_1, \beta_2)\); or
2. \(\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)\) and \(\alpha_1 < \beta_1\); or
3. \(\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)\) and \(\alpha_1 = \beta_1\) and \(\alpha_2 < \beta_2\)

**Lemma card-arithmetic.2.** \((\alpha \times \alpha, \preccurlyeq)\) is a well-order, for any ordinal \(\alpha\).

*Proof.* Evidently \(\preccurlyeq\) is connected on \(\alpha \times \alpha\). For suppose that neither \(\langle \alpha_1, \alpha_2 \rangle\) nor \(\langle \beta_1, \beta_2 \rangle\) is \(\preccurlyeq\)-less than the other. Then \(\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)\) and \(\alpha_1 = \beta_1\) and \(\alpha_2 = \beta_2\), so that \(\langle \alpha_1, \alpha_2 \rangle = \langle \beta_1, \beta_2 \rangle\).

To show well-ordering, let \(X \subseteq \alpha \times \alpha\) be non-empty. Since \(\alpha\) is an ordinal, some \(\delta\) is the least member of \(\{\max(\gamma_1, \gamma_2) : \langle \gamma_1, \gamma_2 \rangle \in X\}\). Now discard all pairs from \(\{\langle \gamma_1, \gamma_2 \rangle \in X : \max(\gamma_1, \gamma_2) = \delta\}\) except those with least first coordinate; from among these, the pair with least second coordinate is the \(\preccurlyeq\)-least element of \(X\). \(\square\)

Now for a teensy, simple observation:

**Proposition card-arithmetic.3.** If \(\alpha \approx \beta\), then \(\alpha \times \alpha \approx \beta \times \beta\).

*Proof.* Just let \(f : \alpha \to \beta\) induce \(\langle \gamma_1, \gamma_2 \rangle \mapsto \langle f(\gamma_1), f(\gamma_2) \rangle\). \(\square\)

And now we will put all this to work, in proving a crucial lemma:

**Lemma card-arithmetic.4.** \(\alpha \approx \alpha \times \alpha\), for any infinite ordinal \(\alpha\)

*Proof.* For reductio, let \(\alpha\) be the least infinite ordinal for which this is false. \(\_?\) shows that \(\omega \approx \omega \times \omega\), so \(\omega \in \alpha\). Moreover, \(\alpha\) is a cardinal: suppose otherwise, for reductio; then \(|\alpha| \in \alpha\), so that \(|\alpha| \approx |\alpha| \times |\alpha|\), by hypothesis; and \(|\alpha| \approx \alpha\) by definition; so that \(\alpha \approx \alpha \times \alpha\) by **Proposition card-arithmetic.3**.

Now, for each \(\langle \gamma_1, \gamma_2 \rangle \in \alpha \times \alpha\), consider the segment:

\[\text{Seg}(\gamma_1, \gamma_2) = \{\langle \delta_1, \delta_2 \rangle \in \alpha \times \alpha : \langle \delta_1, \delta_2 \rangle \preccurlyeq \langle \gamma_1, \gamma_2 \rangle\}\]
Letting $\gamma = \max(\gamma_1, \gamma_2)$, note that $\langle \gamma_1, \gamma_2 \rangle \triangleleft (\gamma + 1, \gamma + 1)$. So, when $\gamma$ is infinite, observe:

\[
\text{Seg}(\gamma_1, \gamma_2) \preceq ((\gamma + 1) \cdot (\gamma + 1)) \\
\approx (\gamma \cdot \gamma), \text{ by ?? and Proposition card-arithmetic.3} \\
\approx \gamma, \text{ by the induction hypothesis} \\
\prec \alpha, \text{ since } \alpha \text{ is a cardinal}
\]

So $\text{ord}(\alpha \times \alpha, \triangleleft) \leq \alpha$, and hence $\alpha \times \alpha \preceq \alpha$. Since of course $\alpha \preceq \alpha \times \alpha$, the result follows by Schröder-Bernstein.

Finally, we get to our simplifying result:

**Theorem card-arithmetic.5.** If $a, b$ are infinite cardinals, then:

\[
a \otimes b = a \oplus b = \max(a, b).
\]

**Proof.** Without loss of generality, suppose $a = \max(a, b)$. Then invoking Lemma card-arithmetic.4, $a \otimes a = a \leq a \oplus b \leq a \oplus a \leq a \otimes a$.

Similarly, if $a$ is infinite, an $a$-sized union of $\leq a$-sized sets has size $\leq a$:

**Proposition card-arithmetic.6.** Let $a$ be an infinite cardinal. For each ordinal $\beta \in a$, let $X_\beta$ be a set with $|X_\beta| \leq a$. Then $\left| \bigcup_{\beta \in a} X_\beta \right| \leq a$.

**Proof.** For each $\beta \in a$, fix an injection $f_\beta: X_\beta \to a$. Define an injection $g: \bigcup_{\beta \in a} X_\beta \to a \times a$ by $g(v) = (\beta, f_\beta(v))$, where $v \in X_\beta$ and $v \notin X_\gamma$ for any $\gamma \in \beta$. Now $\bigcup_{\beta \in a} X_\beta \preceq a \times a \approx a$ by Theorem card-arithmetic.5.

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**1How are these “fixed”? See ??.