card-arithmetic.1 ℵ-Fixed Points

sth:card-arithmetic:fix:sec In **??**, we suggested that Replacement stands in need of justification, because it forces the hierarchy to be rather tall. Having done some cardinal arithmetic, we can give a little illustration of the height of the hierarchy.

Evidently $0 < \aleph_0$, and $1 < \aleph_1$, and $2 < \aleph_2 \ldots$ and, indeed, the difference in size only gets *bigger* with every step. So it is tempting to conjecture that $\kappa < \aleph_\kappa$ for every ordinal κ.

But this conjecture is *false*, given **ZFC**. In fact, we can prove that there are $\aleph$-*fixed-points*, i.e., cardinals κ such that $\kappa = \aleph_\kappa$.

sth:card-arithmetic:fix:alephfixed **Proposition card-arithmetic.1.** *There is an* $\aleph$*-fixed-point.*

Proof. Using recursion, define:

$$\begin{aligned} \kappa_0 &= 0 \\ \kappa_{n+1} &= \aleph_{\kappa_n} \\ \kappa &= \bigcup_{n<\omega} \kappa_n \end{aligned}$$

Now κ is a cardinal by **??**. But now:

$$\kappa = \bigcup_{n<\omega} \kappa_{n+1} = \bigcup_{n<\omega} \aleph_{\kappa_n} = \bigcup_{\alpha<\kappa} \aleph_\alpha = \aleph_\kappa$$

□

Boolos once wrote an article about exactly the ℵ-fixed-point we just constructed. After noting the existence of κ, at the start of his article, he said:

> [κ is] a *pretty big* number, by the lights of those with no previous exposure to set theory, so big, it seems to me, that it calls into question the truth of any theory, one of whose assertions is the claim that there are at least κ objects. (Boolos, 2000, p. 257)

And he ultimately concluded his paper by asking:

> [do] we suspect that, however it may have been at the beginning of the story, by the time we have come thus far the wheels are spinning and we are no longer listening to a description of anything that is the case? (Boolos, 2000, p. 268)

If we have, indeed, outrun "anything that is the case", then we must point the finger of blame directly at Replacement. For it is this axiom which allows our proof to work. In which case, one assumes, Boolos would need to revisit the claim he made, a few decades earlier, that Replacement has "no undesirable" consequences (see **??**).

But is the existence of κ so bad? It might help, here, to consider Russell's *Tristram Shandy paradox*. Tristram Shandy documents his life in his diary, but it takes him a year to record a single day. With every passing year, Tristram falls further and further behind: after one year, he has recorded only one day, and has lived 364 days unrecorded days; after two years, he has only recorded two days, and has lived 728 unrecorded days; after three years, he has only recorded three days, and lived 1092 unrecorded days ...[1] Still, if Tristram is *immortal*, Tristram will manage to record every day, for he will record the nth day on the nth year of his life. And so, "at the end of time", Tristram will have a complete diary.

Now: why is this so different from the thought that α is smaller than $\aleph_\alpha$—and indeed, increasingly, desperately smaller—up until κ, at which point, we catch up, and $\kappa = \aleph_\kappa$?

Setting that aside, and assuming we accept **ZFC**, let's close with a little more fun concerning fixed-point constructions. The next three results establish, intuitively, that there is a (non-trivial) point at which the hierarchy is as wide as it is tall:

Proposition card-arithmetic.2. *There is a $\beth$-fixed-point, i.e., a κ such that $\kappa = \beth_\kappa$.* *sth:card-arithmetic:fix:bethfixed*

Proof. As in Proposition card-arithmetic.1, using "$\beth$" in place of "$\aleph$". □

Proposition card-arithmetic.3. *$|V_{\omega+\alpha}| = \beth_\alpha$. If $\omega \cdot \omega \leq \alpha$, then $|V_\alpha| = \beth_\alpha$.* *sth:card-arithmetic:fix:stagesize*

Proof. The first claim holds by a simple transfinite induction. The second claim follows, since if $\omega \cdot \omega \leq \alpha$ then $\omega + \alpha = \alpha$. To establish this, we use facts about ordinal arithmetic from **??**. First note that $\omega \cdot \omega = \omega \cdot (1 + \omega) = (\omega \cdot 1) + (\omega \cdot \omega) = \omega + (\omega \cdot \omega)$. Now if $\omega \cdot \omega \leq \alpha$, i.e., $\alpha = (\omega \cdot \omega) + \beta$ for some β, then $\omega + \alpha = \omega + ((\omega \cdot \omega) + \beta) = (\omega + (\omega \cdot \omega)) + \beta = (\omega \cdot \omega) + \beta = \alpha$. □

Corollary card-arithmetic.4. *There is a κ such that $|V_\kappa| = \kappa$.*

Proof. Let κ be a $\beth$-fixed point, as given by Proposition card-arithmetic.2. Clearly $\omega \cdot \omega < \kappa$. So $|V_\kappa| = \beth_\kappa = \kappa$ by Proposition card-arithmetic.3. □

There are as many stages beneath V_κ as there are elements of V_κ. Intuitively, then, V_κ is as wide as it is tall. This is very Tristram-Shandy-esque: we move from one stage to the next by taking *powersets*, thereby making our hierarchy *much* bigger with each step. But, "in the end", i.e., at stage κ, the hierarchy's width catches up with its height.

One might ask: *How often does the hierarchy's width match its height?* The answer is: *As often as there are ordinals.* But this needs a little explanation.

We define a term τ as follows. For any A, let:

$$\begin{aligned}
\tau_0(A) &= |A| \\
\tau_{n+1}(A) &= \beth_{\tau_n(A)} \\
\tau(A) &= \bigcup_{n<\omega} \tau_n(A)
\end{aligned}$$

[1]Forgetting about leap years.

As in Proposition card-arithmetic.2, $\tau(A)$ is a $\beth$-fixed point for any A, and trivially $|A| < \tau(A)$. So now consider this recursive definition:

$$\begin{aligned} W_0 &= 0 \\ W_{\alpha+1} &= \tau(W_\alpha) \\ W_\alpha &= \bigcup_{\beta<\alpha} W_\beta, \text{ when } \alpha \text{ is a limit} \end{aligned}$$

The construction is defined for all ordinals. Intuitively, then, W is "an injection" from the ordinals to $\beth$-fixed points. And, exactly as before, V_{W_α} is as wide as it is tall, for any α.

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Bibliography

Boolos, George. 2000. Must we believe in set theory? In *Between Logic and Intuition: Essays in Honor of Charles Parsons*, eds. Gila Sher and Richard Tieszen, 257–68. Cambridge: Cambridge University Press.