

card-arithmetic.1 Some Simplification with Cardinal Exponentiation

Whilst defining \triangleleft was a little involved, the upshot is a useful result concerning cardinal addition and multiplication, $??$. Transfinite exponentiation, however, cannot be simplified so straightforwardly. To explain why, we start with a result which extends a familiar pattern from the finitary case (though its proof is at a high level of abstraction):

Proposition card-arithmetic.1. $\mathfrak{a}^{\mathfrak{b} \oplus \mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \otimes \mathfrak{a}^{\mathfrak{c}}$ and $(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b} \otimes \mathfrak{c}}$, for any cardinals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$.

Proof. For the first claim, consider a function $f: (\mathfrak{b} \sqcup \mathfrak{c}) \rightarrow \mathfrak{a}$. Now “split this”, by defining $f_{\mathfrak{b}}(\beta) = f(\beta, 0)$ for each $\beta \in \mathfrak{b}$, and $f_{\mathfrak{c}}(\gamma) = f(\gamma, 1)$ for each $\gamma \in \mathfrak{c}$. The map $f \mapsto (f_{\mathfrak{b}} \times f_{\mathfrak{c}})$ is a bijection ${}^{\mathfrak{b} \sqcup \mathfrak{c}}\mathfrak{a} \rightarrow ({}^{\mathfrak{b}}\mathfrak{a} \times {}^{\mathfrak{c}}\mathfrak{a})$.

For the second claim, consider a function $f: \mathfrak{c} \rightarrow ({}^{\mathfrak{b}}\mathfrak{a})$; so for each $\gamma \in \mathfrak{c}$ we have some function $f(\gamma): \mathfrak{b} \rightarrow \mathfrak{a}$. Now define $f^*(\beta, \gamma) = (f(\gamma))(\beta)$ for each $\langle \beta, \gamma \rangle \in \mathfrak{b} \times \mathfrak{c}$. The map $f \mapsto f^*$ is a bijection ${}^{\mathfrak{c}}({}^{\mathfrak{b}}\mathfrak{a}) \rightarrow {}^{\mathfrak{b} \otimes \mathfrak{c}}\mathfrak{a}$. \square

Now, what we would *like* is an easy way to compute $\mathfrak{a}^{\mathfrak{b}}$ when we are dealing with infinite cardinals. Here is a nice step in this direction:

Proposition card-arithmetic.2. If $2 \leq \mathfrak{a} \leq \mathfrak{b}$ and \mathfrak{b} is infinite, then $\mathfrak{a}^{\mathfrak{b}} = 2^{\mathfrak{b}}$

Proof.

$$\begin{aligned} 2^{\mathfrak{b}} &\leq \mathfrak{a}^{\mathfrak{b}}, \text{ as } 2 \leq \mathfrak{a} \\ &\leq (2^{\mathfrak{a}})^{\mathfrak{b}}, \text{ by } ?? \\ &= 2^{\mathfrak{a} \otimes \mathfrak{b}}, \text{ by Proposition card-arithmetic.1} \\ &= 2^{\mathfrak{b}}, \text{ by } ?? \end{aligned} \quad \square$$

We should not really expect to be able to simplify this any further, since $\mathfrak{b} < 2^{\mathfrak{b}}$ by $??$. However, this does not tell us what to say about $\mathfrak{a}^{\mathfrak{b}}$ when $\mathfrak{b} < \mathfrak{a}$. Of course, if \mathfrak{b} is *finite*, we know what to do.

Proposition card-arithmetic.3. If \mathfrak{a} is infinite and $n \in \omega$ then $\mathfrak{a}^n = \mathfrak{a}$

Proof. $\mathfrak{a}^n = \mathfrak{a} \otimes \mathfrak{a} \otimes \dots \otimes \mathfrak{a} = \mathfrak{a}$, by $??$. \square

Additionally, in some other cases, we can control the size of $\mathfrak{a}^{\mathfrak{b}}$:

Proposition card-arithmetic.4. If $2 \leq \mathfrak{b} < \mathfrak{a} \leq 2^{\mathfrak{b}}$ and \mathfrak{b} is infinite, then $\mathfrak{a}^{\mathfrak{b}} = 2^{\mathfrak{b}}$

Proof. $2^{\mathfrak{b}} \leq \mathfrak{a}^{\mathfrak{b}} \leq (2^{\mathfrak{b}})^{\mathfrak{b}} = 2^{\mathfrak{b} \otimes \mathfrak{b}} = 2^{\mathfrak{b}}$, reasoning as in Proposition card-arithmetic.2. \square

But, beyond this point, things become rather more subtle.

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Bibliography