The previous result hints (correctly) that cardinal exponentiation would be quite easy, if infinite cardinals are guaranteed to “play straightforwardly” with powers of 2, i.e., (by ??) with taking powersets. But we cannot assume that infinite cardinals do play straightforwardly powersets.

To start unpacking this, we introduce some nice notation.

**Definition card-arithmetic.1.** Where \( a^{\oplus} \) is the least cardinal strictly greater than \( a \), we define two infinite sequences:

\[
\begin{align*}
\aleph_0 &= \omega \\
\beth_0 &= \omega \\
\aleph_{\alpha + 1} &= (\aleph_\alpha)^{\oplus} \\
\beth_{\alpha + 1} &= 2^{\beth_\alpha} \\
\aleph_\alpha &= \bigcup_{\beta < \alpha} \aleph_\beta \\
\beth_\alpha &= \bigcup_{\beta < \alpha} \beth_\beta 
\end{align*}
\]

when \( \alpha \) is a limit ordinal.

The definition of \( a^{\oplus} \) is in order, since ?? tells us that, for each cardinal \( a \), there is some cardinal greater than \( a \), and Transfinite Induction guarantees that there is a least cardinal greater than \( a \). The rest of the definition of \( a \) is provided by transfinite recursion.

Cantor introduced this “\( \aleph \)” notation; this is aleph, the first letter in the Hebrew alphabet and the first letter in the Hebrew word for “infinite”. Peirce introduced the “\( \beth \)” notation; this is beth, which is the second letter in the Hebrew alphabet. \(^1\) Now, these notations provide us with infinite cardinals.

**Proposition card-arithmetic.2.** \( \aleph_\alpha \) and \( \beth_\alpha \) are cardinals, for every ordinal \( \alpha \).

**Proof.** Both results hold by a simple transfinite induction. \( \aleph_0 = \beth_0 = \omega \) is a cardinal by ???. Assuming \( \aleph_\alpha \) and \( \beth_\alpha \) are both cardinals, \( \aleph_{\alpha + 1} \) and \( \beth_{\alpha + 1} \) are explicitly defined as cardinals. And the union of a set of cardinals is a cardinal, by ???. \( \square \)

Moreover, every infinite cardinal is an \( \aleph \):

**Proposition card-arithmetic.3.** If \( a \) is an infinite cardinal, then \( a = \aleph_\gamma \) for some unique \( \gamma \).

**Proof.** By transfinite induction on cardinals. For induction, suppose that if \( b < a \) then \( b = \aleph_\gamma \). If \( a = b^{\oplus} \) for some \( b \), then \( a = (\aleph_\gamma)^{\oplus} = \aleph_{\gamma + 1} \). If \( a \) is not the successor of any cardinal, then since cardinals are ordinals
\[
a = \bigcup_{b < a} b = \bigcup_{b < a} \aleph_\gamma \text{, so } a = \aleph_\gamma \text{ where } \gamma = \bigcup_{b < a} \gamma_b . \]
\( \square \)

\(^1\)Peirce used this notation in a letter to Cantor of December 1900. Unfortunately, Peirce also gave a bad argument there that \( \beth_\alpha \) does not exist for \( \alpha \geq \omega \).
Since every infinite cardinal is an $\aleph$, this prompts us to ask: is every infinite cardinal a $\beth$? Certainly if that were the case, then the infinite cardinals would “play straightforwardly” with the operation of taking powersets. Indeed, we would have the following:

Generalized Continuum Hypothesis (GCH). $\aleph_\alpha = \beth_\alpha$, for all $\alpha$.

Moreover, if GCH held, then we could make some considerable simplifications with cardinal exponentiation. In particular, we could show that when $b < a$, the value of $a^b$ is trapped by $a \leq a^b \leq a^{\oplus}$. We could then go on to give precise conditions which determine which of the two possibilities obtains (i.e., whether $a = a^b$ or $a^b = a^{\oplus}$).

But GCH is a hypothesis, not a theorem. In fact, Gödel (1938) proved that if ZFC is consistent, then so is ZFC + GCH. But it later turned out that we can equally add $\neg$GCH to ZFC. Indeed, consider the simplest non-trivial instance of GCH, namely:

Continuum Hypothesis (CH). $\aleph_1 = \beth_1$.

Cohen (1963) proved that if ZFC is consistent then so is ZFC + $\neg$CH. So the Continuum Hypothesis is independent from ZFC.

The Continuum Hypothesis is so-called, since “the continuum” is another name for the real line, $\mathbb{R}$. ?? tells us that $|\mathbb{R}| = \beth_1$. So the Continuum Hypothesis states that there is no cardinal between the cardinality of the natural numbers, $\aleph_0 = \beth_0$, and the cardinality of the continuum, $\beth_1$.

Given the independence of (G)CH from ZFC, what should say about their truth? Well, there is much to say. Indeed, and much fertile recent work in set theory has been directed at investigating these issues. But two very quick points are certainly worth emphasising.

First: it does not immediately follow from these formal independence results that either GCH or CH is indeterminate in truth value. After all, maybe we just need to add more axioms, which strike us as natural, and which will settle the question one way or another. Gödel himself suggested that this was the right response.

Second: the independence of CH from ZFC is certainly striking, but it is certainly not incredible (in the literal sense). The point is simply that, for all ZFC tells us, moving from cardinals to their successors may involve a less blunt tool than simply taking powersets.

With those two observations made, if you want to know more, you will now have to turn to the various philosophers and mathematicians with horses in the race.\(^3\)

\(^2\)The condition is dictated by cofinality.
\(^3\)Though you might want to start by reading Potter (2004, §15.6).
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Bibliography

