card-arithmetic.1 The Continuum Hypothesis

sth:card-arithmetic:ch:sec

The previous result hints (correctly) that cardinal exponentiation would be quite *easy*, if infinite cardinals are guaranteed to "play straightforwardly" with powers of 2, i.e., (by **??**) with taking powersets. But we cannot assume that infinite cardinals *do* play nicely with powersets. This section is dedicated to explaining all of this. (Although, to be honest, it's more of a *gesture* in the direction of something fascinating.)

We will start by introducing some nice notation.

Definition card-arithmetic.1. Where $\mathfrak{a}^{\oplus}$ is the least cardinal strictly greater than $\mathfrak{a}$, we define two infinite sequences:

$$\begin{aligned} \aleph_0 &:= \omega & \beth_0 &:= \omega \\ \aleph_{\alpha+1} &:= (\aleph_\alpha)^{\oplus} & \beth_{\alpha+1} &:= 2^{\beth_\alpha} \\ \aleph_\alpha &:= \bigcup_{\beta<\alpha} \aleph_\beta & \beth_\alpha &:= \bigcup_{\beta<\alpha} \beth_\beta \qquad \text{when } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The definition of $\mathfrak{a}^{\oplus}$ is in order, since **??** tells us that, for each cardinal $\mathfrak{a}$, there is some cardinal greater than $\mathfrak{a}$, and Transfinite Induction guarantees that there is a *least* cardinal greater than $\mathfrak{a}$. The rest of the definition of $\mathfrak{a}$ is provided by transfinite recursion.

Cantor introduced this "$\aleph$" notation; this is *aleph*, the first letter in the Hebrew alphabet and the first letter in the Hebrew word for "infinite". Peirce introduced the "$\beth$" notation; this is *beth*, which is the second letter in the Hebrew alphabet.[1] Now, these notations provide us with infinite cardinals.

Proposition card-arithmetic.2. *Both $\aleph_\alpha$ and $\beth_\alpha$ are cardinals, for every ordinal α.*

Proof. Both results hold by a simple transfinite induction. $\aleph_0 = \beth_0 = \omega$ is a cardinal by **??**. Assuming $\aleph_\alpha$ and $\beth_\alpha$ are both cardinals, $\aleph_{\alpha+1}$ and $\beth_{\alpha+1}$ are explicitly defined as cardinals. And the union of a set of cardinals is a cardinal, by **??**. □

Moreover, every infinite cardinal is an $\aleph$:

Proposition card-arithmetic.3. *If $\mathfrak{a}$ is an infinite cardinal, then $\mathfrak{a} = \aleph_\gamma$ for some γ.*

Proof. By transfinite induction on cardinals. For induction, suppose that if $\mathfrak{b} < \mathfrak{a}$ then $\mathfrak{b} = \aleph_{\gamma_\mathfrak{b}}$. If $\mathfrak{a} = \mathfrak{b}^{\oplus}$ for some $\mathfrak{b}$, then $\mathfrak{a} = \aleph_{\gamma_\mathfrak{b}}^{\oplus} = \aleph_{\gamma_\mathfrak{b}+1}$. If $\mathfrak{a}$ is not the successor of any cardinal, then since cardinals are ordinals $\mathfrak{a} = \bigcup_{\mathfrak{b}<\mathfrak{a}} \mathfrak{b} = \bigcup_{\mathfrak{b}<\mathfrak{a}} \aleph_{\gamma_\mathfrak{b}}$, so $\mathfrak{a} = \aleph_\gamma$ where $\gamma = \bigcup_{\mathfrak{b}<\mathfrak{a}} \gamma_\mathfrak{b}$. □

[1] Peirce used this notation in a letter to Cantor of December 1900. Unfortunately, Peirce also gave a bad argument there that $\beth_\alpha$ does not exist for $\alpha \geq \omega$.

Since every infinite cardinal is an $\aleph$, this prompts us to ask: is every infinite cardinal a $\beth$? Certainly if that *were* the case, then the infinite cardinals would "play straightforwardly" with the operation of taking powersets. Indeed, we would have the following:

General Continuum Hypothesis (GCH). $\aleph_\alpha = \beth_\alpha$, for all α.

Moreover, if GCH held, then we could make some considerable simplifications with cardinal exponentiation. In particular, we could show that when $\mathfrak{b} < \mathfrak{a}$, the value of $\mathfrak{a}^\mathfrak{b}$ is trapped by $\mathfrak{a} \leq \mathfrak{a}^\mathfrak{b} \leq \mathfrak{a}^\oplus$. We could then go on to give precise conditions which determine which of the two possibilities obtains (i.e., whether $\mathfrak{a} = \mathfrak{a}^\mathfrak{b}$ or $\mathfrak{a}^\mathfrak{b} = \mathfrak{a}^\oplus$).[2]

But GCH is a *hypothesis*, not a *theorem*. In fact, Gödel (1938) proved that if **ZFC** is consistent, then so is **ZFC** + GCH. But it later turned out that we can equally add $\neg$GCH to **ZFC**. Indeed, consider the simplest non-trivial *instance* of GCH, namely:

Continuum Hypothesis (CH). $\aleph_1 = \beth_1$.

Cohen (1963) proved that if **ZFC** is consistent then so is **ZFC** + $\neg$CH.

The Continuum Hypothesis is so-called, since "the continuum" is another name for the real line, $\mathbb{R}$. **??** tells us that $|\mathbb{R}| = \beth_1$. So the Continuum Hypothesis states that there is no cardinal between the cardinality of the natural numbers, $\aleph_0 = \beth_0$, and the cardinality of the continuum, $\beth_1$.

Given the *independence* of (G)CH from **ZFC**, what should say about their *truth*? Well, there is *much* to say. Indeed, and much fertile recent work in set theory has been directed at investigating these issues. But two quick points are certainly worth emphasising.

First: it does not *immediately* follow from these formal independence results that either GCH or CH is *indeterminate* in truth value. After all, maybe we just need to add more axioms, which strike us as natural, and which will settle the question one way or another. Gödel himself suggested that this was the right response.

Second: the independence of CH from **ZFC** is certainly *striking*, but it is certainly not *incredible* (in the literal sense). The point is simply that, for all **ZFC** tells us, moving from cardinals to their successors may involve a less blunt tool than simply taking powersets.

With those two observations made, if you want to know more, you will now have to turn to the various philosophers and mathematicians with horses in the race. (Though you may want to start with the very nice discussion in Potter 2004, §15.6.)

[2]The condition is dictated by *cofinality*.

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Bibliography

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