syn.1 Terms and Formulas

sol:syn:frm:sec Like in first-order logic, expressions of second-order logic are built up from a basic vocabulary containing *variables*, *constant symbols*, *predicate symbols* and sometimes *function symbols*. From them, together with logical connectives, quantifiers, and punctuation symbols such as parentheses and commas, *terms* and *formulas* are formed. The difference is that in addition to variables for objects, second-order logic also contains variables for relations and functions, and allows quantification over them. So the logical symbols of second-order logic are those of first-order logic, plus:

1. A denumerable set of second-order relation variables of every arity n: V_0^n, V_1^n, V_2^n, …
2. A denumerable set of second-order function variables: u_0^n, u_1^n, u_2^n, …

Just as we use x, y, z as meta-variables for first-order variables v_i, we'll use X, Y, Z, etc., as metavariables for V_i^n and u, v, etc., as meta-variables for u_i^n.

The non-logical symbols of a second-order language are specified the same way a first-order language is: by listing its constant symbols, function symbols, and predicate symbols explanation

In first-order logic, the identity predicate $=$ is usually included. In first-order logic, the non-logical symbols of a language $\mathcal{L}$ are crucial to allow us to express anything interesting. There are of course sentences that use no non-logical symbols, but with only $=$ it is hard to say anything interesting. In second-order logic, since we have an unlimited supply of relation and function variables, we can say anything we can say in a first-order language even without a special supply of non-logical symbols.

Definition syn.1 (Second-order Terms). The set of *second-order terms* of $\mathcal{L}$, $\mathrm{Trm}^2(\mathcal{L})$, is defined by adding to **??** the clause

1. If u is an n-place function variable and $t_1, \dots, t_n$ are terms, then $u(t_1, \dots, t_n)$ is a term.

So, a second-order term looks just like a first-order term, except that where a first-order term contains a function symbol f_i^n, a second-order term may contain a function variable u_i^n in its place. explanation

Definition syn.2 (Second-order formula). The set of *second-order formulas* $\mathrm{Frm}^2(\mathcal{L})$ of the language $\mathcal{L}$ is defined by adding to **??** the clauses

1. If X is an n-place predicate variable and $t_1, \dots, t_n$ are second-order terms of $\mathcal{L}$, then $X(t_1, \dots, t_n)$ is an atomic formula.
2. If φ is a formula and u is a function variable, then $\forall u\, \varphi$ is a formula.
3. If φ is a formula and X is a predicate variable, then $\forall X\, \varphi$ is a formula.

4. If φ is a formula and u is a function variable, then $\exists u \, \varphi$ is a formula.
5. If φ is a formula and X is a predicate variable, then $\exists X \, \varphi$ is a formula.

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Bibliography