syn.1 Describing Infinite and Enumerable Domains

sol:syn:siz:sec

A set M is (Dedekind) infinite iff there is an injective function $f\colon M \to M$ which is not surjective, i.e., with $\operatorname{dom}(f) \neq M$. In first-order logic, we can consider a one-place function symbol f and say that the function $f^{\mathfrak{M}}$ assigned to it in a structure $\mathfrak{M}$ is injective and $\operatorname{ran}(f) \neq |\mathfrak{M}|$:

$$\forall x\, \forall y\, (f(x) = f(y) \to x = y) \land \exists y\, \forall x\, y \neq f(x).$$

If $\mathfrak{M}$ satisfies this sentence, $f^{\mathfrak{M}} : |\mathfrak{M}| \to |\mathfrak{M}|$ is injective, and so $|\mathfrak{M}|$ must be infinite. If $|\mathfrak{M}|$ is infinite, and hence such a function exists, we can let $f^{\mathfrak{M}}$ be that function and $\mathfrak{M}$ will satisfy the sentence. However, this requires that our language contains the non-logical symbol f we use for this purpose. In second-order logic, we can simply say that such a function *exists*. This no-longer requires f, and we obtain the sentence in pure second-order logic

$$\mathrm{Inf} \equiv \exists u\, (\forall x\, \forall y\, (u(x) = u(y) \to x = y) \land \exists y\, \forall x\, y \neq u(x)).$$

$\mathfrak{M} \vDash \mathrm{Inf}$ iff $|\mathfrak{M}|$ is infinite. We can then define $\mathrm{Fin} \equiv \lnot\mathrm{Inf}$; $\mathfrak{M} \vDash \mathrm{Fin}$ iff $|\mathfrak{M}|$ is finite. No single sentence of pure first-order logic can express that the domain is infinite although an infinite set of them can. There is no set of sentences of pure first-order logic that is satisfied in a structure iff its domain is finite.

Proposition syn.1. *$\mathfrak{M} \vDash \mathrm{Inf}$ iff $|\mathfrak{M}|$ is infinite.*

Proof. $\mathfrak{M} \vDash \mathrm{Inf}$ iff $\mathfrak{M}, s \vDash \forall x\, \forall y\, (u(x) = u(y) \to x = y) \land \exists y\, \forall x\, y \neq u(x)$ for some s. If it does, $s(u)$ is an injective function, and some $y \in |\mathfrak{M}|$ is not in the domain of $s(u)$. Conversely, if there is an injective $f\colon |\mathfrak{M}| \to |\mathfrak{M}|$ with $\operatorname{dom}(f) \neq |\mathfrak{M}|$, then $s(u) = f$ is such a variable assignment. □

A set M is enumerable if there is an enumeration

$$m_0, m_1, m_2, \ldots$$

of its elements (without repetitions but possibly finite). Such an enumeration exists iff there is an element $z \in M$ and a function $f\colon M \to M$ such that z, $f(z)$, $f(f(z))$, …, are all the elements of M. For if the enumeration exists, $z = m_0$ and $f(m_k) = m_{k+1}$ (or $f(m_k) = m_k$ if m_k is the last element of the enumeration) are the requisite element and function. On the other hand, if such a z and f exist, then z, $f(z)$, $f(f(z))$, …, is an enumeration of M, and M is enumerable. We can express the existence of z and f in second-order logic to produce a sentence true in a structure iff the structure is enumerable:

$$\mathrm{Count} \equiv \exists z\, \exists u\, \forall X\, ((X(z) \land \forall x\, (X(x) \to X(u(x)))) \to \forall x\, X(x))$$

Proposition syn.2. *$\mathfrak{M} \vDash \mathrm{Count}$ iff $|\mathfrak{M}|$ is enumerable.*

Proof. Suppose $|\mathfrak{M}|$ is enumerable, and let m_0, m_1, ..., be an enumeration. By removing repetions we can guarantee that no m_k appears twice. Define $f(m_k) = m_{k+1}$ and let $s(z) = m_0$ and $s(u) = f$. We show that

$$\mathfrak{M}, s \vDash \forall X\,((X(z) \land \forall x\,(X(x) \to X(u(x)))) \to \forall x\,X(x))$$

Suppose $s' \sim_X s$ is arbitrary, and let $M = s'(X)$. Suppose further that $\mathfrak{M}, s' \vDash (X(z) \land \forall x\,(X(x) \to X(u(x))))$. Then $s'(z) \in M$ and whenever $x \in M$, also $s'(u)(x) \in M$. In other words, since $s' \sim_X s$, $m_0 \in M$ and if $x \in M$ then $f(x) \in M$, so $m_0 \in M$, $m_1 = f(m_0) \in M$, $m_2 = f(f(m_0)) \in M$, etc. Thus, $M = |\mathfrak{M}|$, and so $\mathfrak{M} \vDash \forall x\,X(x)s'$. Since s' was an arbitrary X-variant of s, we are done: $\mathfrak{M} \vDash \text{Count}$.

Now assume that $\mathfrak{M} \vDash \text{Count}$, i.e.,

$$\mathfrak{M}, s \vDash \forall X\,((X(z) \land \forall x\,(X(x) \to X(u(x)))) \to \forall x\,X(x))$$

for some s. Let $m = s(z)$ and $f = s(u)$ and consider $M = \{m, f(m), f(f(m)), \dots\}$. Let s' be the X-variant of s with $s(X) = M$. Then

$$\mathfrak{M}, s' \vDash (X(z) \land \forall x\,(X(x) \to X(u(x)))) \to \forall x\,X(x)$$

by assumption. Also, $\mathfrak{M}, s' \vDash X(z)$ since $s'(X) = M \ni m = s'(z)$, and also $\mathfrak{M}, s' \vDash \forall x\,(X(x) \to X(u(x)))$ since whenever $x \in M$ also $f(x) \in M$. So, since both antecedent and conditional are satisfied, the consequent must also be: $\mathfrak{M}, s' \vDash \forall x\,X(x)$. But that means that $M = |\mathfrak{M}|$, and so $|\mathfrak{M}|$ is enumerable since M is, by definition. □

Problem syn.1. The sentence $\text{Inf} \land \text{Count}$ is true in all and only denumerable domains. Adjust the definition of Count so that it becomes a different sentence that directly expresses that the domain is denumerable, and prove that it does.

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Bibliography