

## syn.1 Describing Infinite and Enumerable Domains

sol:syn:siz:  
sec

A set  $M$  is (Dedekind) infinite iff there is an **injective** function  $f: M \rightarrow M$  which is not **surjective**, i.e., with  $\text{dom}(f) \neq M$ . In first-order logic, we can consider a one-place **function symbol**  $f$  and say that the function  $f^{\mathfrak{M}}$  assigned to it in a **structure**  $\mathfrak{M}$  is **injective** and  $\text{ran}(f) \neq |\mathfrak{M}|$ :

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y) \wedge \exists y \forall x y \neq f(x).$$

If  $\mathfrak{M}$  satisfies this **sentence**,  $f^{\mathfrak{M}}: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$  is **injective**, and so  $|\mathfrak{M}|$  must be infinite. If  $|\mathfrak{M}|$  is infinite, and hence such a function exists, we can let  $f^{\mathfrak{M}}$  be that function and  $\mathfrak{M}$  will satisfy the **sentence**. However, this requires that our language contains the non-logical symbol  $f$  we use for this purpose. In second-order logic, we can simply say that such a function *exists*. This no-longer requires  $f$ , and we obtain the **sentence** in pure second-order logic

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x)).$$

$\mathfrak{M} \models \text{Inf}$  iff  $|\mathfrak{M}|$  is infinite. We can then define  $\text{Fin} \equiv \neg \text{Inf}$ ;  $\mathfrak{M} \models \text{Fin}$  iff  $|\mathfrak{M}|$  is finite. No single **sentence** of pure first-order logic can express that the **domain** is infinite although an infinite set of them can. There is no set of **sentences** of pure first-order logic that is satisfied in a **structure** iff its domain is finite.

**Proposition syn.1.**  $\mathfrak{M} \models \text{Inf}$  iff  $|\mathfrak{M}|$  is infinite.

*Proof.*  $\mathfrak{M} \models \text{Inf}$  iff  $\mathfrak{M}, s \models \forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x)$  for some  $s$ . If it does,  $s(u)$  is an **injective** function, and some  $y \in |\mathfrak{M}|$  is not in the domain of  $s(u)$ . Conversely, if there is an **injective**  $f: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$  with  $\text{dom}(f) \neq |\mathfrak{M}|$ , then  $s(u) = f$  is such a variable assignment.  $\square$

A set  $M$  is **enumerable** if there is an enumeration

$$m_0, m_1, m_2, \dots$$

of its **elements** (without repetitions but possibly finite). Such an enumeration exists iff there is an **element**  $z \in M$  and a function  $f: M \rightarrow M$  such that  $z, f(z), f(f(z)), \dots$  are all the **elements** of  $M$ . For if the enumeration exists,  $z = m_0$  and  $f(m_k) = m_{k+1}$  (or  $f(m_k) = m_k$  if  $m_k$  is the last **element** of the enumeration) are the requisite **element** and function. On the other hand, if such a  $z$  and  $f$  exist, then  $z, f(z), f(f(z)), \dots$  is an enumeration of  $M$ , and  $M$  is **enumerable**. We can express the existence of  $z$  and  $f$  in second-order logic to produce a **sentence** true in a **structure** iff the **structure** is **enumerable**:

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

**Proposition syn.2.**  $\mathfrak{M} \models \text{Count}$  iff  $|\mathfrak{M}|$  is **enumerable**.

*Proof.* Suppose  $|\mathfrak{M}|$  is **enumerable**, and let  $m_0, m_1, \dots$ , be an enumeration. By removing repetitions we can guarantee that no  $m_k$  appears twice. Define  $f(m_k) = m_{k+1}$  and let  $s(z) = m_0$  and  $s(u) = f$ . We show that

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

Suppose  $M \subseteq |\mathfrak{M}|$  is arbitrary. Suppose further that  $\mathfrak{M}, s[M/X] \models (X(z) \wedge \forall x (X(x) \rightarrow X(u(x))))$ . Then  $s[M/X](z) \in M$  and whenever  $x \in M$ , also  $(s[M/X](u))(x) \in M$ . In other words, since  $s[M/X] \sim_X s$ ,  $m_0 \in M$  and if  $x \in M$  then  $f(x) \in M$ , so  $m_0 \in M$ ,  $m_1 = f(m_0) \in M$ ,  $m_2 = f(f(m_0)) \in M$ , etc. Thus,  $M = |\mathfrak{M}|$ , and so  $\mathfrak{M}, s[M/X] \models \forall x X(x)$ . Since  $M \subseteq |\mathfrak{M}|$  was arbitrary, we are done:  $\mathfrak{M} \models \text{Count}$ .

Now assume that  $\mathfrak{M} \models \text{Count}$ , i.e.,

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

for some  $s$ . Let  $m = s(z)$  and  $f = s(u)$  and consider  $M = \{m, f(m), f(f(m)), \dots\}$ .  $M$  so defined is clearly **enumerable**. Then

$$\mathfrak{M}, s[M/X] \models (X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x)$$

by assumption. Also,  $\mathfrak{M}, s[M/X] \models X(z)$  since  $M \ni m = s[M/X](z)$ , and also  $\mathfrak{M}, s[M/X] \models \forall x (X(x) \rightarrow X(u(x)))$  since whenever  $x \in M$  also  $f(x) \in M$ . So, since both antecedent and conditional are satisfied, the consequent must also be:  $\mathfrak{M}, s[M/X] \models \forall x X(x)$ . But that means that  $M = |\mathfrak{M}|$ , and so  $|\mathfrak{M}|$  is **enumerable** since  $M$  is, by definition.  $\square$

**Problem syn.1.** The **sentence**  $\text{Inf} \wedge \text{Count}$  is true in all and only **denumerable** domains. Adjust the definition of **Count** so that it becomes a different **sentence** that directly expresses that the domain is **denumerable**, and prove that it does.

## Photo Credits

## Bibliography