

## syn.1 Expressive Power

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sec

Quantification over second-order variables is responsible for an immense increase in the expressive power of the language over that of first-order logic. Second-order existential quantification lets us say that functions or relations with certain properties exist. In first-order logic, the only way to do that is to specify a non-logical symbol (i.e., a **function symbol** or **predicate symbol**) for this purpose. Second-order universal quantification lets us say that all subsets of, relations on, or functions from the **domain** to the **domain** have a property. In first-order logic, we can only say that the subsets, relations, or functions assigned to one of the non-logical symbols of the language have a property. And when we say that subsets, relations, functions exist that have a property, or that all of them have it, we can use second-order quantification in specifying this property as well. This lets us define relations not definable in first-order logic, and express properties of the domain not expressible in first-order logic.

explanation

**Definition syn.1.** If  $\mathfrak{M}$  is a **structure** for a language  $\mathcal{L}$ , a relation  $R \subseteq |\mathfrak{M}|^2$  is *definable* in  $\mathcal{L}$  if there is some **formula**  $\varphi_R(x, y)$  with only the variables  $x$  and  $y$  free, such that  $R(a, b)$  holds (i.e.,  $\langle a, b \rangle \in R$ ) iff  $\mathfrak{M}, s \models \varphi_R(x, y)$  for  $s(x) = a$  and  $s(y) = b$ .

**Example syn.2.** In first-order logic we can define the identity relation  $\text{Id}_{|\mathfrak{M}|}$  (i.e.,  $\{\langle a, a \rangle : a \in |\mathfrak{M}|\}$ ) by the formula  $x = y$ . In second-order logic, we can define this relation *without*  $=$ . For if  $a$  and  $b$  are the same **element** of  $|\mathfrak{M}|$ , then they are **elements** of the same subsets of  $|\mathfrak{M}|$  (since sets are determined by their **elements**). Conversely, if  $a$  and  $b$  are different, then they are not **elements** of the same subsets: e.g.,  $a \in \{a\}$  but  $b \notin \{a\}$  if  $a \neq b$ . So “being **elements** of the same subsets of  $|\mathfrak{M}|$ ” is a relation that holds of  $a$  and  $b$  iff  $a = b$ . It is a relation that can be expressed in second-order logic, since we can quantify over all subsets of  $|\mathfrak{M}|$ . Hence, the following **formula** defines  $\text{Id}_{|\mathfrak{M}|}$ :

$$\forall X (X(x) \leftrightarrow X(y))$$

**Problem syn.1.** Show that  $\forall X (X(x) \rightarrow X(y))$  (note:  $\rightarrow$  not  $\leftrightarrow$ !) defines  $\text{Id}_{|\mathfrak{M}|}$ .

**Example syn.3.** If  $R$  is a two-place **predicate symbol**,  $R^{\mathfrak{M}}$  is a two-place relation on  $|\mathfrak{M}|$ . Perhaps somewhat confusingly, we’ll use  $R$  as the **predicate symbol** for  $R$  and for the relation  $R^{\mathfrak{M}}$  itself. The *transitive closure*  $R^*$  of  $R$  is the relation that holds between  $a$  and  $b$  iff for some  $c_1, \dots, c_k$ ,  $R(a, c_1)$ ,  $R(c_1, c_2)$ ,  $\dots$ ,  $R(c_k, b)$  holds. This includes the case if  $k = 0$ , i.e., if  $R(a, b)$  holds, so does  $R^*(a, b)$ . This means that  $R \subseteq R^*$ . In fact,  $R^*$  is the smallest relation that includes  $R$  and that is transitive. We can say in second-order logic that  $X$  is a transitive relation that includes  $R$ :

$$\begin{aligned} \psi_R(X) \equiv & \forall x \forall y (R(x, y) \rightarrow X(x, y)) \wedge \\ & \forall x \forall y \forall z ((X(x, y) \wedge X(y, z)) \rightarrow X(x, z)). \end{aligned}$$

The first conjunct says that  $R \subseteq X$  and the second that  $X$  is transitive.

To say that  $X$  is the smallest such relation is to say that it is itself included in every relation that includes  $R$  and is transitive. So we can define the transitive closure of  $R$  by the **formula**

$$R^*(X) \equiv \psi_R(X) \wedge \forall Y (\psi_R(Y) \rightarrow \forall x \forall y (X(x, y) \rightarrow Y(x, y))).$$

We have  $\mathfrak{M}, s \models R^*(X)$  iff  $s(X) = R^*$ . The transitive closure of  $R$  cannot be expressed in first-order logic.

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## Bibliography