

**Part I**

**Second-order Logic**

This is the beginnings of a part on second-order logic.

# Chapter 1

## Syntax and Semantics

Basic syntax and semantics for SOL covered so far. As a chapter it's too short. Substitution for second-order variables has to be covered to be able to talk about derivation systems for SOL, and there's some subtle issues there.

### 1.1 Introduction

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sec

In first-order logic, we combine the non-logical symbols of a given language, i.e., its **constant symbols**, **function symbols**, and **predicate symbols**, with the logical symbols to express things about first-order **structures**. This is done using the notion of satisfaction, which relates a **structure**  $\mathfrak{M}$ , together with a variable assignment  $s$ , and a **formula**  $\varphi$ :  $\mathfrak{M}, s \models \varphi$  holds iff what  $\varphi$  expresses when its **constant symbols**, **function symbols**, and **predicate symbols** are interpreted as  $\mathfrak{M}$  says, and its free variables are interpreted as  $s$  says, is true. The interpretation of the **identity predicate**  $=$  is built into the definition of  $\mathfrak{M}, s \models \varphi$ , as is the interpretation of  $\forall$  and  $\exists$ . The former is always interpreted as the identity relation on the **domain**  $|\mathfrak{M}|$  of the structure, and the quantifiers are always interpreted as ranging over the entire **domain**. But, crucially, quantification is only allowed over elements of the **domain**, and so only object **variables** are allowed to follow a quantifier.

In second-order logic, both the language and the definition of satisfaction are extended to include free and bound function and predicate variables, and quantification over them. These variables are related to **function symbols** and **predicate symbols** the same way that object variables are related to **constant symbols**. They play the same role in the formation of terms and **formulas** of second-order logic, and quantification over them is handled in a similar way. In the *standard* semantics, the second-order quantifiers range over all possible objects of the right type ( $n$ -place functions from  $|\mathfrak{M}|$  to  $|\mathfrak{M}|$  for function variables,  $n$ -place relations for predicate variables). For

instance, while  $\forall v_0 (P_0^1(v_0) \vee \neg P_0^1(v_0))$  is a formula in both first- and second-order logic, in the latter we can also consider  $\forall V_0^1 \forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$  and  $\exists V_0^1 \forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$ . Since these contain no free variables, they are **sentences** of second-order logic. Here,  $V_0^1$  is a second-order 1-place predicate variable. The allowable interpretations of  $V_0^1$  are the same that we can assign to a 1-place **predicate symbol** like  $P_0^1$ , i.e., subsets of  $|\mathfrak{M}|$ . Quantification over them then amounts to saying that  $\forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$  holds for all ways of assigning a subset of  $|\mathfrak{M}|$  as the value of  $V_0^1$ , or for at least one. Since every set either contains or fails to contain a given object, both are true in any **structure**.

## 1.2 Terms and Formulas

Like in first-order logic, expressions of second-order logic are built up from a basic vocabulary containing *variables*, *constant symbols*, *predicate symbols* and sometimes *function symbols*. From them, together with logical connectives, quantifiers, and punctuation symbols such as parentheses and commas, *terms* and *formulas* are formed. The difference is that in addition to variables for objects, second-order logic also contains variables for relations and functions, and allows quantification over them. So the logical symbols of second-order logic are those of first-order logic, plus:

1. A **denumerable** set of second-order relation **variables** of every arity  $n$ :  $V_0^n, V_1^n, V_2^n, \dots$
2. A **denumerable** set of second-order function **variables**:  $u_0^n, u_1^n, u_2^n, \dots$

Just as we use  $x, y, z$  as meta-variables for first-order variables  $v_i$ , we'll use  $X, Y, Z$ , etc., as metavariables for  $V_i^n$  and  $u, v$ , etc., as meta-variables for  $u_i^n$ .

explanation

The non-logical symbols of a second-order language are specified the same way a first-order language is: by listing its **constant symbols**, **function symbols**, and **predicate symbols**.

In first-order logic, the **identity predicate**  $=$  is usually included. In first-order logic, the non-logical symbols of a language  $\mathcal{L}$  are crucial to allow us to express anything interesting. There are of course **sentences** that use no non-logical symbols, but with only  $=$  it is hard to say anything interesting. In second-order logic, since we have an unlimited supply of relation and function variables, we can say anything we can say in a first-order language even without a special supply of non-logical symbols.

**Definition 1.1 (Second-order Terms).** The set of *second-order terms* of  $\mathcal{L}$ ,  $\text{Trm}^2(\mathcal{L})$ , is defined by adding to ?? the clause

1. If  $u$  is an  $n$ -place function variable and  $t_1, \dots, t_n$  are terms, then  $u(t_1, \dots, t_n)$  is a term.

So, a second-order term looks just like a first-order term, except that where explanation a first-order term contains a function symbol  $f_i^n$ , a second-order term may contain a function variable  $u_i^n$  in its place.

**Definition 1.2 (Second-order formula).** The set of *second-order formulas*  $\text{Frm}^2(\mathcal{L})$  of the language  $\mathcal{L}$  is defined by adding to ?? the clauses

1. If  $X$  is an  $n$ -place predicate variable and  $t_1, \dots, t_n$  are second-order terms of  $\mathcal{L}$ , then  $X(t_1, \dots, t_n)$  is an atomic formula.
2. If  $\varphi$  is a formula and  $u$  is a function variable, then  $\forall u \varphi$  is a formula.
3. If  $\varphi$  is a formula and  $X$  is a predicate variable, then  $\forall X \varphi$  is a formula.
4. If  $\varphi$  is a formula and  $u$  is a function variable, then  $\exists u \varphi$  is a formula.
5. If  $\varphi$  is a formula and  $X$  is a predicate variable, then  $\exists X \varphi$  is a formula.

### 1.3 Satisfaction

sol:syn:sat: sec To define the satisfaction relation  $\mathfrak{M}, s \models \varphi$  for second-order formulas, we have explanation to extend the definitions to cover second-order variables. The notion of a structure is the same for second-order logic as it is for first-order logic. There is only a difference for variable assignments  $s$ : these now must not just provide values for the first-order variables, but also for the second-order variables.

**Definition 1.3 (Variable Assignment).** A *variable assignment*  $s$  for a structure  $\mathfrak{M}$  is a function which maps each

1. object variable  $v_i$  to an element of  $|\mathfrak{M}|$ , i.e.,  $s(v_i) \in |\mathfrak{M}|$
2.  $n$ -place relation variable  $V_i^n$  to an  $n$ -place relation on  $|\mathfrak{M}|$ , i.e.,  $s(V_i^n) \subseteq |\mathfrak{M}|^n$ ;
3.  $n$ -place function variable  $u_i^n$  to an  $n$ -place function from  $|\mathfrak{M}|$  to  $|\mathfrak{M}|$ , i.e.,  $s(u_i^n): |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$ ;

A structure assigns a value to each constant symbol and function symbol, explanation and a second-order variable assignment assigns objects and functions to each object and function variable. Together, they let us assign a value to every term.

**Definition 1.4 (Value of a Term).** If  $t$  is a term of the language  $\mathcal{L}$ ,  $\mathfrak{M}$  is a structure for  $\mathcal{L}$ , and  $s$  is a variable assignment for  $\mathfrak{M}$ , the *value*  $\text{Val}_s^{\mathfrak{M}}(t)$  is defined as for first-order terms, plus the following clause:

$$t \equiv u(t_1, \dots, t_n):$$

$$\text{Val}_s^{\mathfrak{M}}(t) = s(u)(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)).$$

**Definition 1.5 ( $x$ -Variant).** If  $s$  is a **variable assignment** for a **structure**  $\mathfrak{M}$ , then any **variable assignment**  $s'$  for  $\mathfrak{M}$  which differs from  $s$  at most in what it assigns to  $x$  is called an  $x$ -*variant* of  $s$ . If  $s'$  is an  $x$ -variant of  $s$  we write  $s' \sim_x s$ . (Similarly for second-order variables  $X$  or  $u$ .)

**Definition 1.6.** If  $s$  is a **variable assignment** for a **structure**  $\mathfrak{M}$  and  $m \in |\mathfrak{M}|$ , then the assignment  $s[m/x]$  is the **variable assignment** defined by

$$s[m/y] = \begin{cases} m & \text{if } y \equiv x \\ s(y) & \text{otherwise,} \end{cases}$$

If  $X$  is an  $n$ -place relation **variable** and  $M \subseteq |\mathfrak{M}|^n$ , then  $s[M/X]$  is the **variable assignment** defined by

$$s[M/y] = \begin{cases} M & \text{if } y \equiv X \\ s(y) & \text{otherwise.} \end{cases}$$

If  $u$  is an  $n$ -place function **variable** and  $f: |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$ , then  $s[f/u]$  is the **variable assignment** defined by

$$s[f/y] = \begin{cases} f & \text{if } y \equiv u \\ s(y) & \text{otherwise.} \end{cases}$$

In each case,  $y$  may be any first- or second-order **variable**.

**Definition 1.7 (Satisfaction).** For second-order **formulas**  $\varphi$ , the definition of satisfaction is like ?? with the addition of:

1.  $\varphi \equiv X^n(t_1, \dots, t_n)$ :  $\mathfrak{M}, s \models \varphi$  iff  $\langle \text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n) \rangle \in s(X^n)$ .
2.  $\varphi \equiv \forall X \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for every  $M \subseteq |\mathfrak{M}|^n$ ,  $\mathfrak{M}, s[M/X] \models \psi$ .
3.  $\varphi \equiv \exists X \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for at least one  $M \subseteq |\mathfrak{M}|^n$  so that  $\mathfrak{M}, s[M/X] \models \psi$ .
4.  $\varphi \equiv \forall u \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for every  $f: |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$ ,  $\mathfrak{M}, s[f/u] \models \psi$ .
5.  $\varphi \equiv \exists u \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for at least one  $f: |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$  so that  $\mathfrak{M}, s[f/u] \models \psi$ .

**Example 1.8.** Consider the **formula**  $\forall z (X(z) \leftrightarrow \neg Y(z))$ . It contains no second-order quantifiers, but does contain the second-order **variables**  $X$  and  $Y$  (here understood to be one-place). The corresponding first-order **sentence**  $\forall z (P(z) \leftrightarrow \neg R(z))$  says that whatever falls under the interpretation of  $P$  does not fall under the interpretation of  $R$  and vice versa. In a **structure**, the interpretation of a **predicate symbol**  $P$  is given by the interpretation  $P^{\mathfrak{M}}$ . But for second-order **variables** like  $X$  and  $Y$ , the interpretation is provided, not by the **structure** itself, but by a **variable assignment**. Since the second-order **formula** is not

a **sentence** (it includes free **variables**  $X$  and  $Y$ ), it is only satisfied relative to a **structure**  $\mathfrak{M}$  together with a **variable** assignment  $s$ .

$\mathfrak{M}, s \models \forall z (X(z) \leftrightarrow \neg Y(z))$  whenever the **elements** of  $s(X)$  are not **elements** of  $s(Y)$ , and vice versa, i.e., iff  $s(Y) = |\mathfrak{M}| \setminus s(X)$ . For instance, take  $|\mathfrak{M}| = \{1, 2, 3\}$ . Since no **predicate symbols**, **function symbols**, or **constant symbols** are involved, the domain of  $\mathfrak{M}$  is all that is relevant. Now for  $s_1(X) = \{1, 2\}$  and  $s_1(Y) = \{3\}$ , we have  $\mathfrak{M}, s_1 \models \forall z (X(z) \leftrightarrow \neg Y(z))$ .

By contrast, if we have  $s_2(X) = \{1, 2\}$  and  $s_2(Y) = \{2, 3\}$ ,  $\mathfrak{M}, s_2 \not\models \forall z (X(z) \leftrightarrow \neg Y(z))$ . That's because  $\mathfrak{M}, s_2[2/z] \models X(z)$  (since  $2 \in s_2[2/z](X)$ ) but  $\mathfrak{M}, s_2[2/z] \not\models \neg Y(z)$  (since also  $2 \in s_2[2/z](Y)$ ).

**Example 1.9.**  $\mathfrak{M}, s \models \exists Y (\exists y Y(y) \wedge \forall z (X(z) \leftrightarrow \neg Y(z)))$  if there is an  $N \subseteq |\mathfrak{M}|$  such that  $\mathfrak{M}, s[N/Y] \models (\exists y Y(y) \wedge \forall z (X(z) \leftrightarrow \neg Y(z)))$ . And that is the case for any  $N \neq \emptyset$  (so that  $\mathfrak{M}, s[N/Y] \models \exists y Y(y)$ ) and, as in the previous example,  $M = |\mathfrak{M}| \setminus s(X)$ . In other words,  $\mathfrak{M}, s \models \exists Y (\exists y Y(y) \wedge \forall z (X(z) \leftrightarrow \neg Y(z)))$  iff  $|\mathfrak{M}| \setminus s(X)$  is non-empty, i.e.,  $s(X) \neq |\mathfrak{M}|$ . So, the **formula** is satisfied, e.g., if  $|\mathfrak{M}| = \{1, 2, 3\}$  and  $s(X) = \{1, 2\}$ , but not if  $s(X) = \{1, 2, 3\} = |\mathfrak{M}|$ .

Since the **formula** is not satisfied whenever  $s(X) = |\mathfrak{M}|$ , the **sentence**

$$\forall X \exists Y (\exists y Y(y) \wedge \forall z (X(z) \leftrightarrow \neg Y(z)))$$

is never satisfied: For any **structure**  $\mathfrak{M}$ , the assignment  $s(X) = |\mathfrak{M}|$  will make the **sentence** false. On the other hand, the sentence

$$\exists X \exists Y (\exists y Y(y) \wedge \forall z (X(z) \leftrightarrow \neg Y(z)))$$

is satisfied relative to any assignment  $s$ , since we can always find  $M \subseteq |\mathfrak{M}|$  but  $M \neq |\mathfrak{M}|$  (e.g.,  $M = \emptyset$ ).

**Example 1.10.** The second-order **sentence**  $\forall X \forall y X(y)$  says that every 1-place relation, i.e., every property, holds of every object. That is clearly never true, since in every  $\mathfrak{M}$ , for a variable assignment  $s$  with  $s(X) = \emptyset$ , and  $s(y) = a \in |\mathfrak{M}|$  we have  $\mathfrak{M}, s \not\models X(y)$ . This means that  $\varphi \rightarrow \forall X \forall y X(y)$  is equivalent in second-order logic to  $\neg\varphi$ , that is:  $\mathfrak{M} \models \varphi \rightarrow \forall X \forall y X(y)$  iff  $\mathfrak{M} \models \neg\varphi$ . In other words, in second-order logic we can define  $\neg$  using  $\forall$  and  $\rightarrow$ .

**Problem 1.1.** Show that in second-order logic  $\forall$  and  $\rightarrow$  can define the other connectives:

1. Prove that in second-order logic  $\varphi \wedge \psi$  is equivalent to  $\forall X (\varphi \rightarrow (\psi \rightarrow \forall x X(x)) \rightarrow \forall x X(x))$ .
2. Find a second-order formula using only  $\forall$  and  $\rightarrow$  equivalent to  $\varphi \vee \psi$ .

## 1.4 Semantic Notions

The central logical notions of *validity*, *entailment*, and *satisfiability* are defined the same way for second-order logic as they are for first-order logic, except that the underlying satisfaction relation is now that for second-order **formulas**. A second-order **sentence**, of course, is a **formula** in which all variables, including predicate and function variables, are bound.

**Definition 1.11 (Validity).** A sentence  $\varphi$  is *valid*,  $\models \varphi$ , iff  $\mathfrak{M} \models \varphi$  for every **structure**  $\mathfrak{M}$ .

**Definition 1.12 (Entailment).** A set of sentences  $\Gamma$  *entails* a sentence  $\varphi$ ,  $\Gamma \models \varphi$ , iff for every **structure**  $\mathfrak{M}$  with  $\mathfrak{M} \models \Gamma$ ,  $\mathfrak{M} \models \varphi$ .

**Definition 1.13 (Satisfiability).** A set of sentences  $\Gamma$  is *satisfiable* if  $\mathfrak{M} \models \Gamma$  for some **structure**  $\mathfrak{M}$ . If  $\Gamma$  is not satisfiable it is called *unsatisfiable*.

## 1.5 Expressive Power

explanation

Quantification over second-order variables is responsible for an immense increase in the expressive power of the language over that of first-order logic. Second-order existential quantification lets us say that functions or relations with certain properties exist. In first-order logic, the only way to do that is to specify a non-logical symbol (i.e., a **function symbol** or **predicate symbol**) for this purpose. Second-order universal quantification lets us say that all subsets of, relations on, or functions from the **domain** to the **domain** have a property. In first-order logic, we can only say that the subsets, relations, or functions assigned to one of the non-logical symbols of the language have a property. And when we say that subsets, relations, functions exist that have a property, or that all of them have it, we can use second-order quantification in specifying this property as well. This lets us define relations not definable in first-order logic, and express properties of the domain not expressible in first-order logic.

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**Definition 1.14.** If  $\mathfrak{M}$  is a **structure** for a language  $\mathcal{L}$ , a relation  $R \subseteq |\mathfrak{M}|^2$  is *definable* in  $\mathcal{L}$  if there is some **formula**  $\varphi_R(x, y)$  with only the variables  $x$  and  $y$  free, such that  $R(a, b)$  holds (i.e.,  $\langle a, b \rangle \in R$ ) iff  $\mathfrak{M}, s \models \varphi_R(x, y)$  for  $s(x) = a$  and  $s(y) = b$ .

**Example 1.15.** In first-order logic we can define the identity relation  $\text{Id}_{|\mathfrak{M}|}$  (i.e.,  $\{\langle a, a \rangle : a \in |\mathfrak{M}|\}$ ) by the formula  $x = y$ . In second-order logic, we can define this relation *without*  $=$ . For if  $a$  and  $b$  are the same **element** of  $|\mathfrak{M}|$ , then they are **elements** of the same subsets of  $|\mathfrak{M}|$  (since sets are determined by their **elements**). Conversely, if  $a$  and  $b$  are different, then they are not **elements** of the same subsets: e.g.,  $a \in \{a\}$  but  $b \notin \{a\}$  if  $a \neq b$ . So “being **elements** of the same subsets of  $|\mathfrak{M}|$ ” is a relation that holds of  $a$  and  $b$  iff  $a = b$ . It is a relation that can be expressed in second-order logic, since we can quantify over all subsets of  $|\mathfrak{M}|$ . Hence, the following **formula** defines  $\text{Id}_{|\mathfrak{M}|}$ :

$$\forall X (X(x) \leftrightarrow X(y))$$



**Problem 1.2.** Show that  $\forall X (X(x) \rightarrow X(y))$  (note:  $\rightarrow$  not  $\leftrightarrow$ !) defines  $\text{Id}_{|\mathfrak{M}|}$ .

**Example 1.16.** If  $R$  is a two-place **predicate symbol**,  $R^{\mathfrak{M}}$  is a two-place relation on  $|\mathfrak{M}|$ . Perhaps somewhat confusingly, we'll use  $R$  as the **predicate symbol** for  $R$  and for the relation  $R^{\mathfrak{M}}$  itself. The *transitive closure*  $R^*$  of  $R$  is the relation that holds between  $a$  and  $b$  iff for some  $c_1, \dots, c_k$ ,  $R(a, c_1)$ ,  $R(c_1, c_2), \dots, R(c_k, b)$  holds. This includes the case if  $k = 0$ , i.e., if  $R(a, b)$  holds, so does  $R^*(a, b)$ . This means that  $R \subseteq R^*$ . In fact,  $R^*$  is the smallest relation that includes  $R$  and that is transitive. We can say in second-order logic that  $X$  is a transitive relation that includes  $R$ :

$$\begin{aligned} \psi_R(X) \equiv \forall x \forall y (R(x, y) \rightarrow X(x, y)) \wedge \\ \forall x \forall y \forall z ((X(x, y) \wedge X(y, z)) \rightarrow X(x, z)). \end{aligned}$$

The first conjunct says that  $R \subseteq X$  and the second that  $X$  is transitive.

To say that  $X$  is the smallest such relation is to say that it is itself included in every relation that includes  $R$  and is transitive. So we can define the transitive closure of  $R$  by the **formula**

$$R^*(X) \equiv \psi_R(X) \wedge \forall Y (\psi_R(Y) \rightarrow \forall x \forall y (X(x, y) \rightarrow Y(x, y))).$$

We have  $\mathfrak{M}, s \models R^*(X)$  iff  $s(X) = R^*$ . The transitive closure of  $R$  cannot be expressed in first-order logic.

## 1.6 Describing Infinite and Enumerable Domains

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sec A set  $M$  is (Dedekind) infinite iff there is an **injective** function  $f: M \rightarrow M$  which is not **surjective**, i.e., with  $\text{dom}(f) \neq M$ . In first-order logic, we can consider a one-place **function symbol**  $f$  and say that the function  $f^{\mathfrak{M}}$  assigned to it in a **structure**  $\mathfrak{M}$  is **injective** and  $\text{ran}(f) \neq |\mathfrak{M}|$ :

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y) \wedge \exists y \forall x y \neq f(x).$$

If  $\mathfrak{M}$  satisfies this **sentence**,  $f^{\mathfrak{M}}: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$  is **injective**, and so  $|\mathfrak{M}|$  must be infinite. If  $|\mathfrak{M}|$  is infinite, and hence such a function exists, we can let  $f^{\mathfrak{M}}$  be that function and  $\mathfrak{M}$  will satisfy the **sentence**. However, this requires that our language contains the non-logical symbol  $f$  we use for this purpose. In second-order logic, we can simply say that such a function *exists*. This no-longer requires  $f$ , and we obtain the **sentence** in pure second-order logic

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x)).$$

$\mathfrak{M} \models \text{Inf}$  iff  $|\mathfrak{M}|$  is infinite. We can then define  $\text{Fin} \equiv \neg \text{Inf}$ ;  $\mathfrak{M} \models \text{Fin}$  iff  $|\mathfrak{M}|$  is finite. No single **sentence** of pure first-order logic can express that the **domain** is infinite although an infinite set of them can. There is no set of **sentences** of pure first-order logic that is satisfied in a **structure** iff its domain is finite.

**Proposition 1.17.**  $\mathfrak{M} \models \text{Inf}$  iff  $|\mathfrak{M}|$  is infinite.

*Proof.*  $\mathfrak{M} \models \text{Inf}$  iff  $\mathfrak{M}, s \models \forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x)$  for some  $s$ . If it does,  $s(u)$  is an injective function, and some  $y \in |\mathfrak{M}|$  is not in the domain of  $s(u)$ . Conversely, if there is an injective  $f: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$  with  $\text{dom}(f) \neq |\mathfrak{M}|$ , then  $s(u) = f$  is such a variable assignment.  $\square$

A set  $M$  is enumerable if there is an enumeration

$$m_0, m_1, m_2, \dots$$

of its elements (without repetitions but possibly finite). Such an enumeration exists iff there is an element  $z \in M$  and a function  $f: M \rightarrow M$  such that  $z, f(z), f(f(z)), \dots$ , are all the elements of  $M$ . For if the enumeration exists,  $z = m_0$  and  $f(m_k) = m_{k+1}$  (or  $f(m_k) = m_k$  if  $m_k$  is the last element of the enumeration) are the requisite element and function. On the other hand, if such a  $z$  and  $f$  exist, then  $z, f(z), f(f(z)), \dots$ , is an enumeration of  $M$ , and  $M$  is enumerable. We can express the existence of  $z$  and  $f$  in second-order logic to produce a sentence true in a structure iff the structure is enumerable:

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

**Proposition 1.18.**  $\mathfrak{M} \models \text{Count}$  iff  $|\mathfrak{M}|$  is enumerable.

*Proof.* Suppose  $|\mathfrak{M}|$  is enumerable, and let  $m_0, m_1, \dots$ , be an enumeration. By removing repetitions we can guarantee that no  $m_k$  appears twice. Define  $f(m_k) = m_{k+1}$  and let  $s(z) = m_0$  and  $s(u) = f$ . We show that

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

Suppose  $M \subseteq |\mathfrak{M}|$  is arbitrary. Suppose further that  $\mathfrak{M}, s[M/X] \models (X(z) \wedge \forall x (X(x) \rightarrow X(u(x))))$ . Then  $s[M/X](z) \in M$  and whenever  $x \in M$ , also  $(s[M/X](u))(x) \in M$ . In other words, since  $s[M/X] \sim_X s$ ,  $m_0 \in M$  and if  $x \in M$  then  $f(x) \in M$ , so  $m_0 \in M$ ,  $m_1 = f(m_0) \in M$ ,  $m_2 = f(f(m_0)) \in M$ , etc. Thus,  $M = |\mathfrak{M}|$ , and so  $\mathfrak{M}, s[M/X] \models \forall x X(x)$ . Since  $M \subseteq |\mathfrak{M}|$  was arbitrary, we are done:  $\mathfrak{M} \models \text{Count}$ .

Now assume that  $\mathfrak{M} \models \text{Count}$ , i.e.,

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

for some  $s$ . Let  $m = s(z)$  and  $f = s(u)$  and consider  $M = \{m, f(m), f(f(m)), \dots\}$ .  $M$  so defined is clearly enumerable. Then

$$\mathfrak{M}, s[M/X] \models (X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x)$$

by assumption. Also,  $\mathfrak{M}, s[M/X] \models X(z)$  since  $M \ni m = s[M/X](z)$ , and also  $\mathfrak{M}, s[M/X] \models \forall x (X(x) \rightarrow X(u(x)))$  since whenever  $x \in M$  also  $f(x) \in M$ . So, since both antecedent and conditional are satisfied, the consequent must also be:  $\mathfrak{M}, s[M/X] \models \forall x X(x)$ . But that means that  $M = |\mathfrak{M}|$ , and so  $|\mathfrak{M}|$  is enumerable since  $M$  is, by definition.  $\square$

**Problem 1.3.** The sentence  $\text{Inf} \wedge \text{Count}$  is true in all and only denumerable domains. Adjust the definition of  $\text{Count}$  so that it becomes a different sentence that directly expresses that the domain is denumerable, and prove that it does.

## Chapter 2

# Metatheory of Second-order Logic

### 2.1 Introduction

First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a **derivability** relation  $\vdash$  with the property that for any set of **sentences**  $\Gamma$  and **sentence**  $Q$ ,  $\Gamma \vDash \varphi$  iff  $\Gamma \vdash \varphi$ . This means in particular that the validities of first-order logic are **computably enumerable**. There is a computable function  $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L})$  such that the values of  $f$  are all and only the valid **sentences** of  $\mathcal{L}$ . This is so because **derivations** can be enumerated, and those that **derive** a single **sentence** are then mapped to that **sentence**. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we'll show that second-order logic is not only undecidable, but its validities are not even **computably enumerable**. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

sol:met:int:  
sec

First-order logic also has two more properties: it is compact (if every finite subset of a set  $\Gamma$  of **sentences** is satisfiable,  $\Gamma$  itself is satisfiable) and the Löwenheim–Skolem Theorem holds for it (if  $\Gamma$  has an infinite model it has a **denumerable** model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of **domains** that first-order logic cannot.

### 2.2 Second-order Arithmetic

sol:met:spa:  
sec

Recall that the theory **PA** of Peano arithmetic includes the eight axioms of **Q**,

$$\begin{aligned}
& \forall x x' \neq 0 \\
& \forall x \forall y (x' = y' \rightarrow x = y) \\
& \forall x (x = 0 \vee \exists y x = y') \\
& \forall x (x + 0) = x \\
& \forall x \forall y (x + y') = (x + y)' \\
& \forall x (x \times 0) = 0 \\
& \forall x \forall y (x \times y') = ((x \times y) + x) \\
& \forall x \forall y (x < y \leftrightarrow \exists z (z' + x) = y)
\end{aligned}$$

plus all sentences of the form

$$(\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \varphi(x).$$

The latter is a “schema,” i.e., a pattern that generates infinitely many **sentences** of the language of arithmetic, one for each **formula**  $\varphi(x)$ . We call this schema the (first-order) *axiom schema of induction*. In *second-order* Peano arithmetic **PA<sup>2</sup>**, induction can be stated as a single sentence. **PA<sup>2</sup>** consists of the first eight axioms above plus the (second-order) *induction axiom*:

$$\forall X (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).$$

It says that if a subset  $X$  of the **domain** contains  $0^{\mathfrak{M}}$  and with any  $x \in |\mathfrak{M}|$  also contains  $r^{\mathfrak{M}}(x)$  (i.e., it is “closed under successor”) it contains everything in the **domain** (i.e.,  $X = |\mathfrak{M}|$ ).

The induction axiom guarantees that any **structure** satisfying it contains only those **elements** of  $|\mathfrak{M}|$  the axioms require to be there, i.e., the values of  $\bar{n}$  for  $n \in \mathbb{N}$ . A model of **PA<sup>2</sup>** contains no non-standard numbers.

*sol.met.spa:*  
*thm:sol-pa-standard*

**Theorem 2.1.** *If  $\mathfrak{M} \models \mathbf{PA}^2$  then  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ .*

*Proof.* Let  $N = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ , and suppose  $\mathfrak{M} \models \mathbf{PA}^2$ . Of course, for any  $n \in \mathbb{N}$ ,  $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$ , so  $N \subseteq |\mathfrak{M}|$ .

Now for inclusion in the other direction. Consider a variable assignment  $s$  with  $s(X) = N$ . By assumption,

$$\begin{aligned}
\mathfrak{M} & \models \forall X (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x), \text{ thus} \\
\mathfrak{M}, s & \models (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).
\end{aligned}$$

Consider the antecedent of this conditional.  $\text{Val}^{\mathfrak{M}}(0) \in N$ , and so  $\mathfrak{M}, s \models X(0)$ . The second conjunct,  $\forall x (X(x) \rightarrow X(x'))$  is also satisfied. For suppose  $x \in N$ . By definition of  $N$ ,  $x = \text{Val}^{\mathfrak{M}}(\bar{n})$  for some  $n$ . That gives  $r^{\mathfrak{M}}(x) = \text{Val}^{\mathfrak{M}}(\overline{n+1}) \in N$ . So,  $r^{\mathfrak{M}}(x) \in N$ .

We have that  $\mathfrak{M}, s \models X(0) \wedge \forall x (X(x) \rightarrow X(x'))$ . Consequently,  $\mathfrak{M}, s \models \forall x X(x)$ . But that means that for every  $x \in |\mathfrak{M}|$  we have  $x \in s(X) = N$ . So,  $|\mathfrak{M}| \subseteq N$ .  $\square$

**Corollary 2.2.** *Any two models of  $\mathbf{PA}^2$  are isomorphic.*

*sol:met:spa:  
cor:sol-pa-categorical*

*Proof.* By [Theorem 2.1](#), the domain of any model of  $\mathbf{PA}^2$  is exhausted by  $\text{Val}^{\mathfrak{M}}(\bar{n})$ . Any such model is also a model of  $\mathbf{Q}$ . By ??, any such model is standard, i.e., isomorphic to  $\mathfrak{N}$ .  $\square$

Above we defined  $\mathbf{PA}^2$  as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the *first two* of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

**Proposition 2.3.** *Let  $\mathbf{PA}^{2\ddagger}$  be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then  $\leq$ ,  $+$ , and  $\times$  are definable in  $\mathbf{PA}^{2\ddagger}$ .*

*sol:met:spa:  
prop:sol-pa-definable*

*Proof.* To show that  $\leq$  is definable, we have to find a formula  $\varphi_{\leq}(x, y)$  such that  $\mathfrak{N} \models \varphi_{\leq}(\bar{n}, \bar{m})$  iff  $n \leq m$ . Consider the formula

$$\psi(x, Y) \equiv Y(x) \wedge \forall y (Y(y) \rightarrow Y(y'))$$

Clearly,  $\psi(\bar{n}, Y)$  is satisfied by a set  $Y \subseteq \mathbb{N}$  iff  $\{m : n \leq m\} \subseteq Y$ , so we can take  $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$ .

To see that addition is definable observe that  $k+l = m$  iff there is a function  $u$  such that  $u(0) = k$ ,  $u(n') = u(n)'$  for all  $n$ , and  $m = u(l)$ . We can use this equivalence to define addition in  $\mathbf{PA}^{2\ddagger}$  by the following formula:

$$\varphi_+(x, y, z) \equiv \exists u (u(0) = x \wedge \forall w u(x') = u(x)' \wedge u(y) = z)$$

It should be clear that  $\mathfrak{N} \models \varphi_+(\bar{k}, \bar{l}, \bar{m})$  iff  $k + l = m$ .  $\square$

**Problem 2.1.** Complete the proof of [Proposition 2.3](#).

## 2.3 Second-order Logic is not Axiomatizable

**Theorem 2.4.** *Second-order logic is undecidable.*

*sol:met:nax:  
sec  
sol:met:nax:  
thm:sol-undecidable*

*Proof.* A first-order [sentence](#) is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable.  $\square$

**Theorem 2.5.** *There is no sound and complete [derivation](#) system for second-order logic.*

*sol:met:nax:  
cor:sol-not-axiomatizable*

*Proof.* Let  $\varphi$  be a [sentence](#) in the language of arithmetic.  $\mathfrak{N} \models \varphi$  iff  $\mathbf{PA}^2 \models \varphi$ . Let  $P$  be the conjunction of the nine axioms of  $\mathbf{PA}^2$ .  $\mathbf{PA}^2 \models \varphi$  iff  $\models P \rightarrow \varphi$ , i.e.,  $\mathfrak{M} \models P \rightarrow \varphi$ . Now consider the [sentence](#)  $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$  resulting by replacing  $0$  by  $z$ ,  $'$  by the one-place function variable  $u$ ,  $+$  and  $\times$  by the two-place function-variables  $u'$  and  $u''$ , respectively, and  $<$  by the two-place

relation variable  $L$  and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff  $\mathbf{PA}^2 \models \varphi$  iff  $\mathfrak{N} \models \varphi$ . Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration  $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L}_A)$  of the sentences true in  $\mathfrak{N}$ . This function would be representable in  $\mathbf{Q}$  by some first-order formula  $\psi_f(x, y)$ . Then the formula  $\exists x \psi_f(x, y)$  would define the set of true first-order sentences of  $\mathfrak{N}$ , contradicting Tarski's Theorem.  $\square$

## 2.4 Second-order Logic is not Compact

sol:met:com:  
sec

Call a set of sentences  $\Gamma$  *finitely satisfiable* if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences  $\Gamma$  is finitely satisfiable, it is satisfiable. This property is called *compactness*. It has an equivalent version involving entailment: if  $\Gamma \models \varphi$ , then already  $\Gamma_0 \models \varphi$  for some finite subset  $\Gamma_0 \subseteq \Gamma$ . In this version it is an immediate corollary of the completeness theorem: for if  $\Gamma \models \varphi$ , by completeness  $\Gamma \vdash \varphi$ . But a derivation can only make use of finitely many sentences of  $\Gamma$ .

explanation

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some  $\varphi$  without a finite subset entailing  $\varphi$ .

sol:met:com:  
thm:sol-undecidable

**Theorem 2.6.** *Second-order logic is not compact.*

*Proof.* Recall that

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x))$$

is satisfied in a structure iff its domain is infinite. Let  $\varphi^{\geq n}$  be a sentence that asserts that the domain has at least  $n$  elements, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \dots \exists x_n (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \dots \wedge x_{n-1} \neq x_n).$$

Consider the set of sentences

$$\Gamma = \{\neg \text{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \dots\}.$$

It is finitely satisfiable, since for any finite subset  $\Gamma_0 \subseteq \Gamma$  there is some  $k$  so that  $\varphi^{\geq k} \in \Gamma_0$  but no  $\varphi^{\geq n} \in \Gamma_0$  for  $n > k$ . If  $|\mathfrak{M}|$  has  $k$  elements,  $\mathfrak{M} \models \Gamma_0$ . But,  $\Gamma$  is not satisfiable: if  $\mathfrak{M} \models \neg \text{Inf}$ ,  $|\mathfrak{M}|$  must be finite, say, of size  $k$ . Then  $\mathfrak{M} \not\models \varphi^{\geq k+1}$ .  $\square$

**Problem 2.2.** Give an example of a set  $\Gamma$  and a sentence  $\varphi$  so that  $\Gamma \models \varphi$  but for every finite subset  $\Gamma_0 \subseteq \Gamma$ ,  $\Gamma_0 \not\models \varphi$ .

## 2.5 The Löwenheim–Skolem Theorem Fails for Second-order Logic

**explanation** The (Downward) Löwenheim–Skolem Theorem states that every set of **sentences** with an infinite model has an **enumerable** model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of **sentences**, and that model is **enumerable**. There is also an Upward Löwenheim–Skolem Theorem, which guarantees that if a set of **sentences** has a **denumerable** model it also has a **non-enumerable** model. Both theorems fail in second-order logic. **sol:met:lst:sec**

**Theorem 2.7.** *The Löwenheim–Skolem Theorem fails for second-order logic: There are **sentences** with infinite models but no **enumerable** models.* **sol:met:lst:thm:sol-no-ls**

*Proof.* Recall that

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

is true in a **structure**  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  is **enumerable**, so  $\neg\text{Count}$  is true in  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  is **non-enumerable**. There are such **structures**—take any **non-enumerable** set as the **domain**, e.g.,  $\wp(\mathbb{N})$  or  $\mathbb{R}$ . So  $\neg\text{Count}$  has infinite models but no **enumerable** models. □

**Theorem 2.8.** *There are **sentences** with **denumerable** but no **non-enumerable** models.*

*Proof.*  $\text{Count} \wedge \text{Inf}$  is true in  $\mathbb{N}$  but not in any **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}|$  **non-enumerable**. □



## Chapter 3

# Second-order Logic and Set Theory

This section deals with coding powersets and the continuum in second-order logic. The results are stated but proofs have yet to be filled in. There are no problems yet—and the definitions and results themselves may have problems. Use with caution and report anything that’s false or unclear.

### 3.1 Introduction

sol:set:int:  
sec Since second-order logic can quantify over subsets of the domain as well as functions, it is to be expected that some amount, at least, of set theory can be carried out in second-order logic. By “carry out,” we mean that it is possible to express set theoretic properties and statements in second-order logic, and is possible without any special, non-logical vocabulary for sets (e.g., the membership **predicate symbol** of set theory). For instance, we can define unions and intersections of sets and the subset relationship, but also compare the sizes of sets, and state results such as Cantor’s Theorem.

### 3.2 Comparing Sets

sol:set:cmp:  
sec **Proposition 3.1.** *The formula  $\forall x (X(x) \rightarrow Y(x))$  defines the subset relation, i.e.,  $\mathfrak{M}, s \models \forall x (X(x) \rightarrow Y(x))$  iff  $s(X) \subseteq s(Y)$ .*

**Proposition 3.2.** *The formula  $\forall x (X(x) \leftrightarrow Y(x))$  defines the identity relation on sets, i.e.,  $\mathfrak{M}, s \models \forall x (X(x) \leftrightarrow Y(x))$  iff  $s(X) = s(Y)$ .*

**Proposition 3.3.** *The formula  $\exists x X(x)$  defines the property of being non-empty, i.e.,  $\mathfrak{M}, s \models \exists x X(x)$  iff  $s(X) \neq \emptyset$ .*

A set  $X$  is no larger than a set  $Y$ ,  $X \preceq Y$ , iff there is an **injective** function  $f: X \rightarrow Y$ . Since we can express that a function is injective, and also that its values for arguments in  $X$  are in  $Y$ , we can also define the relation of being no larger than on subsets of the domain.

**Proposition 3.4.** *The formula*

$$\exists u (\forall x (X(x) \rightarrow Y(u(x))) \wedge \forall x \forall y (u(x) = u(y) \rightarrow x = y))$$

*defines the relation of being no larger than.*

Two sets are the same size, or “equinumerous,”  $X \approx Y$ , iff there is a **bijective** function  $f: X \rightarrow Y$ .

**Proposition 3.5.** *The formula*

$$\begin{aligned} \exists u (\forall x (X(x) \rightarrow Y(u(x))) \wedge \\ \forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \\ \forall y (Y(y) \rightarrow \exists x (X(x) \wedge y = u(x)))) \end{aligned}$$

*defines the relation of being equinumerous with.*

We will abbreviate these **formulas**, respectively, as  $X \subseteq Y$ ,  $X = Y$ ,  $X \neq \emptyset$ ,  $X \preceq Y$ , and  $X \approx Y$ . (This may be slightly confusing, since we use the same notation when we speak informally about sets  $X$  and  $Y$ —but here the notation is an abbreviation for **formulas** in second-order logic involving one-place relation variables  $X$  and  $Y$ .)

**Proposition 3.6.** *The **sentence**  $\forall X \forall Y ((X \preceq Y \wedge Y \preceq X) \rightarrow X \approx Y)$  is valid.*

*Proof.* The **sentence** is satisfied in a **structure**  $\mathfrak{M}$  if, for any subsets  $X \subseteq |\mathfrak{M}|$  and  $Y \subseteq |\mathfrak{M}|$ , if  $X \preceq Y$  and  $Y \preceq X$  then  $X \approx Y$ . But this holds for *any* sets  $X$  and  $Y$ —it is the Schröder-Bernstein Theorem.  $\square$

### 3.3 Cardinalities of Sets

explanation Just as we can express that the domain is finite or infinite, **enumerable** or **non-enumerable**, we can define the property of a subset of  $|\mathfrak{M}|$  being finite or infinite, **enumerable** or **non-enumerable**. sol:set:crd: sec

**Proposition 3.7.** *The formula  $\text{Inf}(X) \equiv$*

$$\begin{aligned} \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \\ \exists y (X(y) \wedge \forall x (X(x) \rightarrow y \neq u(x)))) \end{aligned}$$

*is satisfied with respect to a variable assignment  $s$  iff  $s(X)$  is infinite.*

**Proposition 3.8.** *The formula  $\text{Count}(X) \equiv$*

$$\begin{aligned} \exists z \exists u (X(z) \wedge \forall x (X(x) \rightarrow X(u(x))) \wedge \\ \forall Y ((Y(z) \wedge \forall x (Y(x) \rightarrow Y(u(x)))) \rightarrow X = Y)) \end{aligned}$$

*is satisfied with respect to a variable assignment  $s$  iff  $s(X)$  is enumerable.*

We know from Cantor’s Theorem that there are **non-enumerable** sets, and in fact, that there are infinitely many different levels of infinite sizes. Set theory develops an entire arithmetic of sizes of sets, and assigns infinite cardinal numbers to sets. The natural numbers serve as the cardinal numbers measuring the sizes of finite sets. The cardinality of **denumerable** sets is the first infinite cardinality, called  $\aleph_0$  (“aleph-nought” or “aleph-zero”). The next infinite size is  $\aleph_1$ . It is the smallest size a set can be without being countable (i.e., of size  $\aleph_0$ ). We can define “ $X$  has size  $\aleph_0$ ” as  $\text{Aleph}_0(X) \leftrightarrow \text{Inf}(X) \wedge \text{Count}(X)$ .  $X$  has size  $\aleph_1$  iff all its subsets are finite or have size  $\aleph_0$ , but is not itself of size  $\aleph_0$ . Hence we can express this by the formula  $\text{Aleph}_1(X) \equiv \forall Y (Y \subseteq X \rightarrow (\neg \text{Inf}(Y) \vee \text{Aleph}_0(Y))) \wedge \neg \text{Aleph}_0(X)$ . Being of size  $\aleph_2$  is defined similarly, etc.

There is one size of special interest, the so-called cardinality of the continuum. It is the size of  $\wp(\mathbb{N})$ , or, equivalently, the size of  $\mathbb{R}$ . That a set is the size of the continuum can also be expressed in second-order logic, but requires a bit more work.

### 3.4 The Power of the Continuum

sol:set:pow: sec In second-order logic we can quantify over subsets of the **domain**, but not over explanation sets of subsets of the **domain**. To do this directly, we would need *third-order* logic. For instance, if we wanted to state Cantor’s Theorem that there is no **injective** function from the power set of a set to the set itself, we might try to formulate it as “for every set  $X$ , and every set  $P$ , if  $P$  is the power set of  $X$ , then not  $P \preceq X$ ”. And to say that  $P$  is the power set of  $X$  would require formalizing that the **elements** of  $P$  are all and only the subsets of  $X$ , so something like  $\forall Y (P(Y) \leftrightarrow Y \subseteq X)$ . The problem lies in  $P(Y)$ : that is not **a formula** of second-order logic, since only terms can be arguments to one-place relation variables like  $P$ .

We can, however, *simulate* quantification over sets of sets, if the domain is large enough. The idea is to make use of the fact that two-place relations  $R$  relates **elements** of the **domain** to **elements** of the **domain**. Given such an  $R$ , we can collect all the **elements** to which some  $x$  is  $R$ -related:  $\{y \in |\mathfrak{M}| : R(x, y)\}$  is the set “coded by”  $x$ . Conversely, if  $Z \subseteq \wp(|\mathfrak{M}|)$  is some collection of subsets of  $|\mathfrak{M}|$ , and there are at least as many **elements** of  $|\mathfrak{M}|$  as there are sets in  $Z$ , then there is also a relation  $R \subseteq |\mathfrak{M}|^2$  such that every  $Y \in Z$  is coded by some  $x$  using  $R$ .

**Definition 3.9.** If  $R \subseteq |\mathfrak{M}|^2$ , then  $x$  *R-codes*  $\{y \in |\mathfrak{M}| : R(x, y)\}$ .

If an element  $x \in |\mathfrak{M}|$   $R$ -codes a set  $Z \subseteq |\mathfrak{M}|$ , then a set  $Y \subseteq |\mathfrak{M}|$  codes a set of sets, namely the sets coded by the elements of  $Y$ . So a set  $Y$  can  $R$ -code  $\wp(X)$ . It does so iff for every  $Z \subseteq X$ , some  $x \in Y$   $R$ -codes  $Z$ , and every  $x \in Y$   $R$ -codes a  $Z \subseteq X$ .

**Proposition 3.10.** *The formula*

$$\text{Codes}(x, R, Z) \equiv \forall y (Z(y) \leftrightarrow R(x, y))$$

*expresses that  $s(x)$   $s(R)$ -codes  $s(Z)$ . The formula*

$$\begin{aligned} \text{Pow}(Y, R, X) \equiv \\ \forall Z (Z \subseteq X \rightarrow \exists x (Y(x) \wedge \text{Codes}(x, R, Z))) \wedge \\ \forall x (Y(x) \rightarrow \forall Z (\text{Codes}(x, R, Z) \rightarrow Z \subseteq X)) \end{aligned}$$

*expresses that  $s(Y)$   $s(R)$ -codes the power set of  $s(X)$ , i.e., the elements of  $s(Y)$   $s(R)$ -code exactly the subsets of  $s(X)$ .*

explanation

With this trick, we can express statements about the power set by quantifying over the codes of subsets rather than the subsets themselves. For instance, Cantor's Theorem can now be expressed by saying that there is no injective function from the domain of any relation that codes the power set of  $X$  to  $X$  itself.

**Proposition 3.11.** *The sentence*

$$\begin{aligned} \forall X \forall Y \forall R (\text{Pow}(Y, R, X) \rightarrow \\ \neg \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \\ \forall x (Y(x) \rightarrow X(u(x)))))) \end{aligned}$$

*is valid.*

explanation

The power set of a denumerable set is non-enumerable, and so its cardinality is larger than that of any denumerable set (which is  $\aleph_0$ ). The size of  $\wp(\mathbb{N})$  is called the ‘‘power of the continuum,’’ since it is the same size as the points on the real number line,  $\mathbb{R}$ . If the domain is large enough to code the power set of a denumerable set, we can express that a set is the size of the continuum by saying that it is equinumerous with any set  $Y$  that codes the power set of set  $X$  of size  $\aleph_0$ . (If the domain is not large enough, i.e., it contains no subset equinumerous with  $\mathbb{R}$ , then there can also be no relation that codes  $\wp(X)$ .)

**Proposition 3.12.** *If  $\mathbb{R} \preceq |\mathfrak{M}|$ , then the formula*

$$\begin{aligned} \text{Cont}(Y) \equiv \exists X \exists R ((\text{Aleph}_0(X) \wedge \text{Pow}(Y, R, X)) \wedge \\ \forall x \forall y ((Y(x) \wedge Y(y) \wedge \forall z R(x, z) \leftrightarrow R(y, z)) \rightarrow x = y)) \end{aligned}$$

*expresses that  $s(Y) \approx \mathbb{R}$ .*

*Proof.*  $\text{Pow}(Y, R, X)$  expresses that  $s(Y)$   $s(R)$ -codes the power set of  $s(X)$ , which  $\text{Aleph}_0(X)$  says is countable. So  $s(Y)$  is at least as large as the power of the continuum, although it may be larger (if multiple elements of  $s(Y)$  code the same subset of  $X$ ). This is ruled out by the last conjunct, which requires the association between elements of  $s(Y)$  and subsets of  $s(Z)$  via  $s(R)$  to be **injective**.  $\square$

**Proposition 3.13.**  $|\mathfrak{M}| \approx \mathbb{R}$  iff

$$\mathfrak{M} \models \exists X \exists Y \exists R (\text{Aleph}_0(X) \wedge \text{Pow}(Y, R, X) \wedge \\ \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \\ \forall y (Y(y) \rightarrow \exists x y = u(x)))).$$

The Continuum Hypothesis is the statement that the size of the continuum [explanation](#) is the first **non-enumerable** cardinality, i.e., that  $\wp(\mathbb{N})$  has size  $\aleph_1$ .

**Proposition 3.14.** *The Continuum Hypothesis is true iff*

$$\text{CH} \equiv \forall X (\text{Aleph}_1(X) \leftrightarrow \text{Cont}(X))$$

*is valid.*

Note that it isn't true that  $\neg\text{CH}$  is valid iff the Continuum Hypothesis is false. In **an enumerable** domain, there are no subsets of size  $\aleph_1$  and also no subsets of the size of the continuum, so CH is always true in **an enumerable** domain. However, we can give a different sentence that is valid iff the Continuum Hypothesis is false:

**Proposition 3.15.** *The Continuum Hypothesis is false iff*

$$\text{NCH} \equiv \forall X (\text{Cont}(X) \rightarrow \exists Y (Y \subseteq X \wedge \neg\text{Count}(Y) \wedge \neg X \approx Y))$$

*is valid.*

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# Bibliography