met.1  Second-order Logic is not Axiomatizable

Theorem met.1.  Second-order logic is undecidable.

Proof.  A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. □

Theorem met.2.  There is no sound and complete derivation system for second-order logic.

Proof.  Let φ be a sentence in the language of arithmetic.  N ⊨ φ iff PA^2 ⊨ φ.  Let P be the conjunction of the nine axioms of PA^2.  PA^2 ⊨ φ iff P ⊨ φ, i.e., \[ \forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi') \] resulting by replacing \( 0 \) by \( z \), \( 1 \) by the one-place function variable \( u \), \( + \) and \( \times \) by the two-place function-variables \( u' \) and \( u'' \), respectively, and \( < \) by the two-place relation variable \( L \) and universally quantifying.  It is a valid sentence of pure second-order logic iff the original sentence was valid iff PA^2 ⊨ φ iff \( \forall \models \varphi \).  Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration \( f : N \rightarrow Sent(L_A) \) of the sentences true in \( \mathfrak{N} \).  This function would be representable in Q by some first-order formula \( \psi_f(x, y) \).  Then the formula \( \exists x \psi_f(x, y) \) would define the set of true first-order sentences of \( \mathfrak{N} \), contradicting Tarski’s Theorem. □

Photo Credits

Bibliography