met.1 Second-order Logic is not Axiomatizable

sol:met:nax:sec

sol:met:nax:thm:sol-undecidable

Theorem met.1. *Second-order logic is undecidable.*

Proof. A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. □

sol:met:nax:cor:sol-not-axiomatizable

Theorem met.2. *There is no sound and complete derivation system for second-order logic.*

Proof. Let φ be a sentence in the language of arithmetic. $\mathfrak{N} \vDash \varphi$ iff $\mathbf{PA^2} \vDash \varphi$. Let P be the conjunction of the nine axioms of $\mathbf{PA^2}$. $\mathbf{PA^2} \vDash \varphi$ iff $\vDash P \to \varphi$, i.e., $\mathfrak{M} \vDash P \to \varphi$. Now consider the sentence $\forall z\, \forall u\, \forall u'\, \forall u''\, \forall L\, (P' \to \varphi')$ resulting by replacing $\mathtt{0}$ by z, $\prime$ by the one-place function variable u, $+$ and $\times$ by the two-place function-variables u' and u'', respectively, and $<$ by the two-place relation variable L and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff $\mathbf{PA^2} \vDash \varphi$ iff $\mathfrak{N} \vDash \varphi$. Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration $f\colon \mathbb{N} \to \mathrm{Sent}(\mathcal{L}_A)$ of the sentences true in $\mathfrak{N}$. This function would be representable in $\mathbf{Q}$ by some first-order formula $\psi_f(x, y)$. Then the formula $\exists x\, \psi_f(x, y)$ would define the set of true first-order sentences of $\mathfrak{N}$, contradicting Tarski's Theorem. □

Photo Credits

Bibliography