

## met.1 Second-order Logic is not Axiomatizable

sol:met:nax:  
sec  
sol:met:nax:  
thm:sol-undecidable

**Theorem met.1.** *Second-order logic is undecidable.*

*Proof.* A first-order **sentence** is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable.  $\square$

sol:met:nax:  
cor:sol-not-axiomatizable

**Theorem met.2.** *There is no sound and complete proof system for second-order logic.*

*Proof.* Let  $\varphi$  be a **sentence** in the language of arithmetic.  $\mathfrak{N} \models \varphi$  iff  $\mathbf{PA}^2 \models \varphi$ . Let  $P$  be the conjunction of the nine axioms of  $\mathbf{PA}^2$ .  $\mathbf{PA}^2 \models \varphi$  iff  $P \rightarrow \varphi$ , i.e.,  $\mathfrak{N} \models P \rightarrow \varphi$ . Now consider the **sentence**  $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$  resulting by replacing  $0$  by  $z$ ,  $+$  by the one-place function variable  $u$ ,  $+$  and  $\times$  by the two-place function-variables  $u'$  and  $u''$ , respectively, and  $<$  by the two-place relation variable  $L$  and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff  $\mathbf{PA}^2 \models \varphi$  iff  $\mathfrak{N} \models \varphi$ . Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration  $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L}_A)$  of the **sentences** true in  $\mathfrak{N}$ . This function would be representable in  $\mathbf{Q}$  by some first-order formula  $\psi_f(x, y)$ . Then the **formula**  $\exists x \psi_f(x, y)$  would define the set of true first-order **sentences** of  $\mathfrak{N}$ , contradicting Tarski's Theorem.  $\square$

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## Bibliography