

## met.1 Second-order Arithmetic

sol:met:spa: Recall that the theory **PA** of Peano arithmetic includes the eight axioms of **Q**,  
sec

$$\begin{aligned}
 & \forall x \, x' \neq 0 \\
 & \forall x \, \forall y \, (x' = y' \rightarrow x = y) \\
 & \forall x \, (x = 0 \vee \exists y \, x = y') \\
 & \forall x \, (x + 0) = x \\
 & \forall x \, \forall y \, (x + y') = (x + y)' \\
 & \forall x \, (x \times 0) = 0 \\
 & \forall x \, \forall y \, (x \times y') = ((x \times y) + x) \\
 & \forall x \, \forall y \, (x < y \leftrightarrow \exists z \, (z' + x) = y)
 \end{aligned}$$

plus all sentences of the form

$$(\varphi(0) \wedge \forall x \, (\varphi(x) \rightarrow \varphi(x')))) \rightarrow \forall x \, \varphi(x).$$

The latter is a “schema,” i.e., a pattern that generates infinitely many **sentences** of the language of arithmetic, one for each **formula**  $\varphi(x)$ . We call this schema the (first-order) *axiom schema of induction*. In *second-order* Peano arithmetic **PA<sup>2</sup>**, induction can be stated as a single sentence. **PA<sup>2</sup>** consists of the first eight axioms above plus the (second-order) *induction axiom*:

$$\forall X \, (X(0) \wedge \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x).$$

It says that if a subset  $X$  of the **domain** contains  $0^{\mathfrak{M}}$  and with any  $x \in |\mathfrak{M}|$  also contains  $\iota^{\mathfrak{M}}(x)$  (i.e., it is “closed under successor”) it contains everything in the **domain** (i.e.,  $X = |\mathfrak{M}|$ ).

The induction axiom guarantees that any **structure** satisfying it contains only those **elements** of  $|\mathfrak{M}|$  the axioms require to be there, i.e., the values of  $\bar{n}$  for  $n \in \mathbb{N}$ . A model of **PA<sup>2</sup>** contains no non-standard numbers.

sol:met:spa: **Theorem met.1.** *If  $\mathfrak{M} \models \mathbf{PA}^2$  then  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ .*  
thm:sol-pa-standard

*Proof.* Let  $N = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ , and suppose  $\mathfrak{M} \models \mathbf{PA}^2$ . Of course, for any  $n \in \mathbb{N}$ ,  $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$ , so  $N \subseteq |\mathfrak{M}|$ .

Now for inclusion in the other direction. Consider a variable assignment  $s$  with  $s(X) = N$ . By assumption,

$$\begin{aligned}
 \mathfrak{M} & \models \forall X \, (X(0) \wedge \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x), \text{ thus} \\
 \mathfrak{M}, s & \models (X(0) \wedge \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x).
 \end{aligned}$$

Consider the antecedent of this conditional.  $\text{Val}^{\mathfrak{M}}(0) \in N$ , and so  $\mathfrak{M}, s \models X(0)$ . The second conjunct,  $\forall x \, (X(x) \rightarrow X(x'))$  is also satisfied. For suppose  $x \in N$ . By definition of  $N$ ,  $x = \text{Val}^{\mathfrak{M}}(\bar{n})$  for some  $n$ . That gives  $\iota^{\mathfrak{M}}(x) = \text{Val}^{\mathfrak{M}}(\overline{n+1}) \in N$ . So,  $\iota^{\mathfrak{M}}(x) \in N$ .

We have that  $\mathfrak{M}, s \models X(0) \wedge \forall x (X(x) \rightarrow X(x'))$ . Consequently,  $\mathfrak{M}, s \models \forall x X(x)$ . But that means that for every  $x \in |\mathfrak{M}|$  we have  $x \in s(X) = N$ . So,  $|\mathfrak{M}| \subseteq N$ .  $\square$

**Corollary met.2.** *Any two models of  $\mathbf{PA}^2$  are isomorphic.*

*sol:met:spa:  
cor:sol-pa-categorical*

*Proof.* By **Theorem met.1**, the domain of any model of  $\mathbf{PA}^2$  is exhausted by  $\text{Val}^{\mathfrak{M}}(\bar{n})$ . Any such model is also a model of  $\mathbf{Q}$ . By ??, any such model is standard, i.e., isomorphic to  $\mathfrak{N}$ .  $\square$

Above we defined  $\mathbf{PA}^2$  as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the *first two* of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

**Proposition met.3.** *Let  $\mathbf{PA}^{2\ddagger}$  be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then  $\leq$ ,  $+$ , and  $\times$  are definable in  $\mathbf{PA}^{2\ddagger}$ .*

*sol:met:spa:  
prop:sol-pa-definable*

*Proof.* To show that  $\leq$  is definable, we have to find a formula  $\varphi_{\leq}(x, y)$  such that  $\mathfrak{N} \models \varphi_{\leq}(\bar{n}, \bar{m})$  iff  $n < m$ . Consider the formula

$$\psi(x, Y) \equiv Y(x) \wedge \forall y (Y(y) \rightarrow Y(y'))$$

Clearly,  $\psi(\bar{n}, Y)$  is satisfied by a set  $Y \subseteq \mathbb{N}$  iff  $\{m : n \leq m\} \subseteq Y$ , so we can take  $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$ .  $\square$

**Problem met.1.** Complete the proof of **Proposition met.3**.

**Corollary met.4.**  $\mathfrak{M} \models \mathbf{PA}^2$  iff  $\mathfrak{M} \models \mathbf{PA}^{2\ddagger}$ .

*Proof.* Immediate from **Proposition met.3**.  $\square$

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## Bibliography