

met.1 Second-order Arithmetic

sol:met:spa: Recall that the theory **PA** of Peano arithmetic includes the eight axioms of **Q**,
sec

$$\begin{aligned}
& \forall x x' \neq 0 \\
& \forall x \forall y (x' = y' \rightarrow x = y) \\
& \forall x (x = 0 \vee \exists y x = y') \\
& \forall x (x + 0) = x \\
& \forall x \forall y (x + y') = (x + y)' \\
& \forall x (x \times 0) = 0 \\
& \forall x \forall y (x \times y') = ((x \times y) + x) \\
& \forall x \forall y (x < y \leftrightarrow \exists z (z' + x) = y)
\end{aligned}$$

plus all sentences of the form

$$(\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \varphi(x).$$

The latter is a “schema,” i.e., a pattern that generates infinitely many **sentences** of the language of arithmetic, one for each **formula** $\varphi(x)$. We call this schema the (first-order) *axiom schema of induction*. In *second-order* Peano arithmetic **PA²**, induction can be stated as a single sentence. **PA²** consists of the first eight axioms above plus the (second-order) *induction axiom*:

$$\forall X (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).$$

It says that if a subset X of the **domain** contains $0^{\mathfrak{M}}$ and with any $x \in |\mathfrak{M}|$ also contains $\iota^{\mathfrak{M}}(x)$ (i.e., it is “closed under successor”) it contains everything in the **domain** (i.e., $X = |\mathfrak{M}|$).

The induction axiom guarantees that any **structure** satisfying it contains only those **elements** of $|\mathfrak{M}|$ the axioms require to be there, i.e., the values of \bar{n} for $n \in \mathbb{N}$. A model of **PA²** contains no non-standard numbers.

sol:met:spa: **Theorem met.1.** *If $\mathfrak{M} \models \mathbf{PA}^2$ then $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$.*
thm:sol-pa-standard

Proof. Let $N = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$, and suppose $\mathfrak{M} \models \mathbf{PA}^2$. Of course, for any $n \in \mathbb{N}$, $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$, so $N \subseteq |\mathfrak{M}|$.

Now for inclusion in the other direction. Consider a variable assignment s with $s(X) = N$. By assumption,

$$\begin{aligned}
\mathfrak{M} & \models \forall X (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x), \text{ thus} \\
\mathfrak{M}, s & \models (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).
\end{aligned}$$

Consider the antecedent of this conditional. $\text{Val}^{\mathfrak{M}}(0) \in N$, and so $\mathfrak{M}, s \models X(0)$. The second conjunct, $\forall x (X(x) \rightarrow X(x'))$ is also satisfied. For suppose $x \in N$. By definition of N , $x = \text{Val}^{\mathfrak{M}}(\bar{n})$ for some n . That gives $\iota^{\mathfrak{M}}(x) = \text{Val}^{\mathfrak{M}}(\overline{n+1}) \in N$. So, $\iota^{\mathfrak{M}}(x) \in N$.

We have that $\mathfrak{M}, s \models X(0) \wedge \forall x (X(x) \rightarrow X(x'))$. Consequently, $\mathfrak{M}, s \models \forall x X(x)$. But that means that for every $x \in |\mathfrak{M}|$ we have $x \in s(X) = N$. So, $|\mathfrak{M}| \subseteq N$. \square

Corollary met.2. *Any two models of \mathbf{PA}^2 are isomorphic.*

*sol:met:spa:
cor:sol-pa-categorical*

Proof. By **Theorem met.1**, the domain of any model of \mathbf{PA}^2 is exhausted by $\text{Val}^{\mathfrak{M}}(\bar{n})$. Any such model is also a model of \mathbf{Q} . By ??, any such model is standard, i.e., isomorphic to \mathfrak{N} . \square

Above we defined \mathbf{PA}^2 as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the *first two* of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

Proposition met.3. *Let $\mathbf{PA}^{2\ddagger}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then \leq , $+$, and \times are definable in $\mathbf{PA}^{2\ddagger}$.*

*sol:met:spa:
prop:sol-pa-definable*

Proof. To show that \leq is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathfrak{N} \models \varphi_{\leq}(\bar{n}, \bar{m})$ iff $n \leq m$. Consider the formula

$$\psi(x, Y) \equiv Y(x) \wedge \forall y (Y(y) \rightarrow Y(y'))$$

Clearly, $\psi(\bar{n}, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$.

To see that addition is definable observe that $k+l = m$ iff there is a function u such that $u(0) = k$, $u(n') = u(n)'$ for all n , and $m = u(l)$. We can use this equivalence to define addition in $\mathbf{PA}^{2\ddagger}$ by the following formula:

$$\varphi_+(x, y, z) \equiv \exists u (u(0) = x \wedge \forall w u(x') = u(x)' \wedge u(y) = z)$$

It should be clear that $\mathfrak{N} \models \varphi_+(\bar{k}, \bar{l}, \bar{m})$ iff $k + l = m$. \square

Problem met.1. Complete the proof of **Proposition met.3**.

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Bibliography