Chapter udf

Metatheory of Second-order Logic

met.1 Introduction

First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a derivability relation $\vdash$ with the property that for any set of sentences $\Gamma$ and sentence $Q$, $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$. This means in particular that the validities of first-order logic are computably enumerable. There is a computable function $f: \mathbb{N} \to \text{Sent}(L)$ such that the values of $f$ are all and only the valid sentences of $L$. This is so because derivations can be enumerated, and those that derive a single sentence are then mapped to that sentence. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we’ll show that second-order logic is not only undecidable, but its validities are not even computably enumerable. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set $\Gamma$ of sentences is satisfiable, $\Gamma$ itself is satisfiable) and the Löwenheim–Skolem Theorem holds for it (if $\Gamma$ has an infinite model it has a denumerable model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of domains that first-order logic cannot.

met.2 Second-order Arithmetic
Recall that the theory \( \text{PA} \) of Peano arithmetic includes the eight axioms of \( \textbf{Q} \),
\begin{align*}
\forall x \, x \neq 0 \\
\forall x \, \forall y \, (x' = y' \rightarrow x = y) \\
\forall x \, (x = 0 \lor \exists y \, x = y') \\
\forall x \, (x + 0) = x \\
\forall x \, \forall y \, (x + y') = (x + y)' \\
\forall x \, (x \times 0) = 0 \\
\forall x \, \forall y \, (x \times y') = ((x \times y) + x) \\
\forall x \, \forall y \, (x < y \leftrightarrow \exists z \, (z' + x) = y)
\end{align*}

plus all sentences of the form

\[(\phi(0) \land \forall x \, (\phi(x) \rightarrow \phi(x'))) \rightarrow \forall x \, \phi(x).\]

The latter is a “schema,” i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula \( \phi(x) \). We call this schema the (first-order) axion schema of induction. In second-order Peano arithmetic \( \text{PA}^2 \), induction can be stated as a single sentence. \( \text{PA}^2 \) consists of the first eight axioms above plus the (second-order) induction axiom:

\[\forall X \, (X(0) \land \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x).\]

It says that if a subset \( X \) of the domain contains \( 0^M \) and with any \( x \in [M] \) also contains \( \rho^M(x) \) (i.e., it is “closed under successor”) it contains everything in the domain (i.e., \( X = |M| \)).

The induction axiom guarantees that any structure satisfying it contains only those elements of \( |M| \) the axioms require to be there, i.e., the values of \( \pi \) for \( n \in \mathbb{N} \). A model of \( \text{PA}^2 \) contains no non-standard numbers.

**Theorem met.1.** If \( M \models \text{PA}^2 \) then \( |M| = \{ \text{Val}^M(\pi) : n \in \mathbb{N} \} \).

**Proof.** Let \( N = \{ \text{Val}^M(\pi) : n \in \mathbb{N} \} \), and suppose \( M \models \text{PA}^2 \). Of course, for any \( n \in \mathbb{N} \), \( \text{Val}^M(\pi) \in |M| \), so \( N \subseteq |M| \).

Now for inclusion in the other direction. Consider a variable assignment \( s \) with \( s(X) = N \). By assumption,
\[M \models \forall X \, (X(0) \land \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x),\]
\[M, s \models (X(0) \land \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x).\]

Consider the antecedent of this conditional. \( \text{Val}^M(\pi) \in N \), and so \( M, s \models X(0) \). The second conjunct, \( \forall x \, (X(x) \rightarrow X(x')) \) is also satisfied. For suppose \( x \in N \). By definition of \( N \), \( x = \text{Val}^M(\pi) \) for some \( n \). That gives \( \rho^M(x) = \text{Val}^M(n + 1) \in N \). So, \( \rho^M(x) \in N \).

We have that \( M, s \models X(0) \land \forall x \, (X(x) \rightarrow X(x')) \). Consequently, \( M, s \models \forall x \, X(x) \). But that means that for every \( x \in |M| \) we have \( x \in s(X) = N \). So, \( |M| \subseteq N \).
Corollary met.2. Any two models of $\mathbf{PA}^2$ are isomorphic.

Proof. By Theorem met.1, the domain of any model of $\mathbf{PA}^2$ is exhausted by $\text{Val}^{\mathcal{M}}(\pi)$. Any such model is also a model of $\mathbf{Q}$. By ???, any such model is standard, i.e., isomorphic to $\mathcal{M}$.

Above we defined $\mathbf{PA}^2$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the first two of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

Proposition met.3. Let $\mathbf{PA}^{2\dagger}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then $\leq$, $+$, and $\times$ are definable in $\mathbf{PA}^{2\dagger}$.

Proof. To show that $\leq$ is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathcal{M} \models \varphi_{\leq}(n, m)$ iff $n \leq m$. Consider the formula

$$\psi(x, Y) \equiv Y(x) \land \forall y (Y(y) \rightarrow Y(y'))$$

Clearly, $\psi(\overline{n}, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m \} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$.

To see that addition is definable observe that $k+l = m$ iff there is a function $u$ such that $u(0) = k$, $u(n') = u(n)'$ for all $n$, and $m = u(l)$. We can use this equivalence to define addition in $\mathbf{PA}^{2\dagger}$ by the following formula:

$$\varphi_+(x, y, z) \equiv \exists u (u(0) = x \land \forall w u(x') = u(x')' \land u(y) = z)$$

It should be clear that $\mathcal{M} \models \varphi_+(\overline{k}, \overline{l}, \overline{m})$ iff $k + l = m$.

Problem met.1. Complete the proof of Proposition met.3.

Second-order Logic is not Axiomatizable

Theorem met.4. Second-order logic is undecidable.

Proof. A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable.

Theorem met.5. There is no sound and complete derivation system for second-order logic.

Proof. Let $\varphi$ be a sentence in the language of arithmetic. $\mathcal{M} \models \varphi$ iff $\mathbf{PA}^2 \models \varphi$. Let $P$ be the conjunction of the nine axioms of $\mathbf{PA}^2$. $\mathbf{PA}^2 \models \varphi$ iff $\models P \rightarrow \varphi$, i.e., $\mathcal{M} \models P \rightarrow \varphi$. Now consider the sentence $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$ resulting by replacing $0$ by $z$, $t$ by the one-place function variable $u$, $+$ and $\times$ by the two-place function-variables $u'$ and $u''$, respectively, and $<$ by the two-place...
relation variable \( L \) and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff \( \mathsf{PA}^2 \models \varphi \) iff \( \mathfrak{N} \models \varphi \). Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration \( f : \mathbb{N} \to \text{Sent}(L_A) \) of the sentences true in \( \mathfrak{N} \). This function would be representable in \( \mathbb{Q} \) by some first-order formula \( \psi(f(x, y)) \). Then the formula \( \exists x \psi_f(x, y) \) would define the set of true first-order sentences of \( \mathfrak{N} \), contradicting Tarski’s Theorem.

**meta.4 Second-order Logic is not Compact**

Call a set of sentences \( \Gamma \) **finitely satisfiable** if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences \( \Gamma \) is finitely satisfiable, it is satisfiable. This property is called **compactness**. It has an equivalent version involving entailment: if \( \Gamma \models \varphi \), then already \( \Gamma_0 \models \varphi \) for some finite subset \( \Gamma_0 \subseteq \Gamma \). In this version it is an immediate corollary of the completeness theorem: for if \( \Gamma \models \varphi \), by completeness \( \Gamma \vdash \varphi \). But a derivation can only make use of finitely many sentences of \( \Gamma \).

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some \( \varphi \) without a finite subset entailing \( \varphi \).

**Theorem meta.6.** Second-order logic is not compact.

**Proof.** Recall that

\[
\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \land \exists y \forall x y \neq u(x))
\]

is satisfied in a structure iff its domain is infinite. Let \( \varphi^2 \) be a sentence that asserts that the domain has at least \( n \) elements, e.g.,

\[
\varphi^2 \equiv \exists x_1 \ldots \exists x_n (x_1 \neq x_2 \land x_1 \neq x_3 \land \cdots \land x_{n-1} \neq x_n).
\]

Consider the set of sentences

\[
\Gamma = \{ \neg \text{Inf}, \varphi^1, \varphi^2, \varphi^3, \ldots \}.
\]

It is finitely satisfiable, since for any finite subset \( \Gamma_0 \subseteq \Gamma \) there is some \( k \) so that \( \varphi^k \in \Gamma \) but no \( \varphi^n \in \Gamma \) for \( n > k \). If \( \mathfrak{M} \) has \( k \) elements, \( \mathfrak{M} \models \Gamma_0 \).

But, \( \Gamma \) is not satisfiable: if \( \mathfrak{M} \models \neg \text{Inf} \), \( \mathfrak{M} \) must be finite, say, of size \( k \). Then \( \mathfrak{M} \not\models \varphi^k+1 \).

**Problem meta.2.** Give an example of a set \( \Gamma \) and a sentence \( \varphi \) so that \( \Gamma \models \varphi \) but for every finite subset \( \Gamma_0 \subseteq \Gamma \), \( \Gamma_0 \not\models \varphi \).
The (Downward) Löwenheim–Skolem Theorem states that every set of sentences with an infinite model has an enumerable model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim–Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic.

**Theorem met.7.** The Löwenheim–Skolem Theorem fails for second-order logic: There are sentences with infinite models but no enumerable models.

**Proof.** Recall that

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \land \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

is true in a structure $\mathcal{M}$ iff $|\mathcal{M}|$ is enumerable, so $\neg\text{Count}$ is true in $\mathcal{M}$ iff $|\mathcal{M}|$ is non-enumerable. There are such structures—take any non-enumerable set as the domain, e.g., $\mathcal{P}(\mathbb{N})$ or $\mathbb{R}$. So $\neg\text{Count}$ has infinite models but no enumerable models.

**Theorem met.8.** There are sentences with denumerable but no non-enumerable models.

**Proof.** $\text{Count} \land \text{Inf}$ is true in $\mathbb{N}$ but not in any structure $\mathcal{M}$ with $|\mathcal{M}|$ non-enumerable.
Bibliography