

## Chapter udf

# Metatheory of Second-order Logic

### met.1 Introduction

sol:met:int:  
sec

First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a **derivability** relation  $\vdash$  with the property that for any set of **sentences**  $\Gamma$  and **sentence**  $Q$ ,  $\Gamma \models \varphi$  iff  $\Gamma \vdash \varphi$ . This means in particular that the validities of first-order logic are **computably enumerable**. There is a computable function  $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L})$  such that the values of  $f$  are all and only the valid **sentences** of  $\mathcal{L}$ . This is so because **derivations** can be enumerated, and those that **derive** a single **sentence** are then mapped to that **sentence**. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we'll show that second-order logic is not only undecidable, but its validities are not even **computably enumerable**. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set  $\Gamma$  of **sentences** is satisfiable,  $\Gamma$  itself is satisfiable) and the Löwenheim-Skolem Theorem holds for it (if  $\Gamma$  has an infinite model it has a **denumerable** model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of **domains** that first-order logic cannot.

### met.2 Second-order Arithmetic

sol:met:spa:  
sec

Recall that the theory **PA** of Peano arithmetic includes the eight axioms of **Q**,

$$\begin{aligned}
&\forall x x' \neq 0 \\
&\forall x \forall y (x' = y' \rightarrow x = y) \\
&\forall x (x = 0 \vee \exists y x = y') \\
&\forall x (x + 0) = x \\
&\forall x \forall y (x + y') = (x + y)' \\
&\forall x (x \times 0) = 0 \\
&\forall x \forall y (x \times y') = ((x \times y) + x) \\
&\forall x \forall y (x < y \leftrightarrow \exists z (z' + x) = y)
\end{aligned}$$

plus all sentences of the form

$$(\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \varphi(x).$$

The latter is a “schema,” i.e., a pattern that generates infinitely many **sentences** of the language of arithmetic, one for each **formula**  $\varphi(x)$ . We call this schema the (first-order) *axiom schema of induction*. In *second-order* Peano arithmetic **PA<sup>2</sup>**, induction can be stated as a single sentence. **PA<sup>2</sup>** consists of the first eight axioms above plus the (second-order) *induction axiom*:

$$\forall X (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).$$

It says that if a subset  $X$  of the **domain** contains  $0^{\mathfrak{M}}$  and with any  $x \in |\mathfrak{M}|$  also contains  $r^{\mathfrak{M}}(x)$  (i.e., it is “closed under successor”) it contains everything in the **domain** (i.e.,  $X = |\mathfrak{M}|$ ).

The induction axiom guarantees that any **structure** satisfying it contains only those **elements** of  $|\mathfrak{M}|$  the axioms require to be there, i.e., the values of  $\bar{n}$  for  $n \in \mathbb{N}$ . A model of **PA<sup>2</sup>** contains no non-standard numbers.

**Theorem met.1.** *If  $\mathfrak{M} \models \mathbf{PA}^2$  then  $|\mathfrak{M}| = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ .*

*sol:met:spa:  
thm:sol-pa-standard*

*Proof.* Let  $N = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$ , and suppose  $\mathfrak{M} \models \mathbf{PA}^2$ . Of course, for any  $n \in \mathbb{N}$ ,  $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$ , so  $N \subseteq |\mathfrak{M}|$ .

Now for inclusion in the other direction. Consider a variable assignment  $s$  with  $s(X) = N$ . By assumption,

$$\begin{aligned}
\mathfrak{M} \models \forall X (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x), \text{ thus} \\
\mathfrak{M}, s \models (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).
\end{aligned}$$

Consider the antecedent of this conditional.  $\text{Val}^{\mathfrak{M}}(0) \in N$ , and so  $\mathfrak{M}, s \models X(0)$ . The second conjunct,  $\forall x (X(x) \rightarrow X(x'))$  is also satisfied. For suppose  $x \in N$ . By definition of  $N$ ,  $x = \text{Val}^{\mathfrak{M}}(\bar{n})$  for some  $n$ . That gives  $r^{\mathfrak{M}}(x) = \text{Val}^{\mathfrak{M}}(\overline{n+1}) \in N$ . So,  $r^{\mathfrak{M}}(x) \in N$ .

We have that  $\mathfrak{M}, s \models X(0) \wedge \forall x (X(x) \rightarrow X(x'))$ . Consequently,  $\mathfrak{M}, s \models \forall x X(x)$ . But that means that for every  $x \in |\mathfrak{M}|$  we have  $x \in s(X) = N$ . So,  $|\mathfrak{M}| \subseteq N$ .  $\square$

*sol:met:spa:*  
*cor:sol-pa-categorical*

**Corollary met.2.** *Any two models of  $\mathbf{PA}^2$  are isomorphic.*

*Proof.* By **Theorem met.1**, the domain of any model of  $\mathbf{PA}^2$  is exhausted by  $\text{Val}^{\mathfrak{M}}(\bar{n})$ . Any such model is also a model of  $\mathbf{Q}$ . By ??, any such model is standard, i.e., isomorphic to  $\mathfrak{N}$ .  $\square$

Above we defined  $\mathbf{PA}^2$  as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the *first two* of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

*sol:met:spa:*  
*prop:sol-pa-definable*

**Proposition met.3.** *Let  $\mathbf{PA}^{2\ddagger}$  be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then  $\leq$ ,  $+$ , and  $\times$  are definable in  $\mathbf{PA}^{2\ddagger}$ .*

*Proof.* To show that  $\leq$  is definable, we have to find a formula  $\varphi_{\leq}(x, y)$  such that  $\mathfrak{N} \models \varphi_{\leq}(\bar{n}, \bar{m})$  iff  $n < m$ . Consider the formula

$$\psi(x, Y) \equiv Y(x) \wedge \forall y (Y(y) \rightarrow Y(y'))$$

Clearly,  $\psi(\bar{n}, Y)$  is satisfied by a set  $Y \subseteq \mathbb{N}$  iff  $\{m : n \leq m\} \subseteq Y$ , so we can take  $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$ .  $\square$

**Problem met.1.** Complete the proof of **Proposition met.3**.

**Corollary met.4.**  $\mathfrak{M} \models \mathbf{PA}^2$  iff  $\mathfrak{M} \models \mathbf{PA}^{2\ddagger}$ .

*Proof.* Immediate from **Proposition met.3**.  $\square$

### met.3 Second-order Logic is not Axiomatizable

*sol:met:nax:*  
*sec*  
*sol:met:nax:*  
*thm:sol-undecidable*

**Theorem met.5.** *Second-order logic is undecidable.*

*Proof.* A first-order **sentence** is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable.  $\square$

*sol:met:nax:*  
*cor:sol-not-axiomatizable*

**Theorem met.6.** *There is no sound and complete proof system for second-order logic.*

*Proof.* Let  $\varphi$  be a **sentence** in the language of arithmetic.  $\mathfrak{N} \models \varphi$  iff  $\mathbf{PA}^2 \models \varphi$ . Let  $P$  be the conjunction of the nine axioms of  $\mathbf{PA}^2$ .  $\mathbf{PA}^2 \models \varphi$  iff  $\models P \rightarrow \varphi$ , i.e.,  $\mathfrak{M} \models P \rightarrow \varphi$ . Now consider the **sentence**  $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$  resulting by replacing  $0$  by  $z$ ,  $'$  by the one-place function variable  $u$ ,  $+$  and  $\times$  by the two-place function-variables  $u'$  and  $u''$ , respectively, and  $<$  by the two-place relation variable  $L$  and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff  $\mathbf{PA}^2 \models \varphi$  iff  $\mathfrak{N} \models \varphi$ . Thus if there were a sound and complete proof system for second-order logic,

we could use it to define a computable enumeration  $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L}_A)$  of the **sentences** true in  $\mathfrak{N}$ . This function would be representable in  $\mathbf{Q}$  by some first-order formula  $\psi_f(x, y)$ . Then the **formula**  $\exists x \psi_f(x, y)$  would define the set of true first-order **sentences** of  $\mathfrak{N}$ , contradicting Tarski's Theorem.  $\square$

## met.4 Second-order Logic is not Compact

**explanation** Call a set of sentences  $\Gamma$  *finitely satisfiable* if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of **sentences**  $\Gamma$  is finitely satisfiable, it is satisfiable. This property is called *compactness*. It has an equivalent version involving entailment: if  $\Gamma \models \varphi$ , then already  $\Gamma_0 \models \varphi$  for some finite subset  $\Gamma_0 \subseteq \Gamma$ . In this version it is an immediate corollary of the completeness theorem: for if  $\Gamma \models \varphi$ , by completeness  $\Gamma \vdash \varphi$ . But a **derivation** can only make use of finitely many **sentences** of  $\Gamma$ .

**sol:met:com:sec**

Compactness is not true for second-order logic. There are sets of second-order **sentences** that are finitely satisfiable but not satisfiable, and that entail some  $\varphi$  without a finite subset entailing  $\varphi$ .

**Theorem met.7.** *Second-order logic is not compact.*

**sol:met:com:thm:sol-undecidable**

*Proof.* Recall that

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x))$$

is satisfied in a **structure** iff its domain is infinite. Let  $\varphi^{\geq n}$  be a sentence that asserts that the domain has at least  $n$  **elements**, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \dots \exists x_n (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \dots \wedge x_{n-1} \neq x_n).$$

Consider the set of **sentences**

$$\Gamma = \{\neg \text{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \dots\}.$$

It is finitely satisfiable, since for any finite subset  $\Gamma_0 \subseteq \Gamma$  there is some  $k$  so that  $\varphi^{\geq k} \in \Gamma_0$  but no  $\varphi^{\geq n} \in \Gamma_0$  for  $n > k$ . If  $|\mathfrak{M}|$  has  $k$  **elements**,  $\mathfrak{M} \models \Gamma_0$ . But,  $\Gamma$  is not satisfiable: if  $\mathfrak{M} \models \neg \text{Inf}$ ,  $|\mathfrak{M}|$  must be finite, say, of size  $k$ . Then  $\mathfrak{M} \not\models \varphi^{\geq k+1}$ .  $\square$

**Problem met.2.** Give an example of a set  $\Gamma$  and a **sentence**  $\varphi$  so that  $\Gamma \models \varphi$  but for every finite subset  $\Gamma_0 \subseteq \Gamma$ ,  $\Gamma_0 \not\models \varphi$ .

## met.5 The Löwenheim-Skolem Theorem Fails for Second-order Logic

**explanation** The (Downward) Löwenheim-Skolem Theorem states that every set of **sentences** with an infinite model has an **enumerable** model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a

**sol:met:lst:sec**

model for any consistent set of **sentences**, and that model is **enumerable**. There is also an Upward Löwenheim-Skolem Theorem, which guarantees that if a set of **sentences** has a **denumerable** model it also has a **non-enumerable** model. Both theorems fail in second-order logic.

*sol:met:lst: thm:sol-no-ls* **Theorem met.8.** *The Löwenheim-Skolem Theorem fails for second-order logic: There are **sentences** with infinite models but no **enumerable** models.*

*Proof.* Recall that

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

is true in a **structure**  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  is **enumerable**, so  $\neg\text{Count}$  is true in  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  is **non-enumerable**. There are such **structures**—take any **non-enumerable** set as the **domain**, e.g.,  $\wp(\mathbb{N})$  or  $\mathbb{R}$ . So  $\neg\text{Count}$  has infinite models but no **enumerable** models.  $\square$

**Theorem met.9.** *There are **sentences** with **denumerable** but no **non-enumerable** models.*

*Proof.*  $\text{Count} \wedge \text{Inf}$  is true in  $\mathbb{N}$  but not in any **structure**  $\mathfrak{M}$  with  $|\mathfrak{M}|$  **non-enumerable**.  $\square$

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# Bibliography