met.1 The Löwenheim–Skolem Theorem Fails for Second-order Logic

sol:met:lst:sec The (Downward) Löwenheim–Skolem Theorem states that every set of sentences with an infinite model has an enumerable model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim–Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic. explanation

sol:met:lst:thm:sol-no-ls **Theorem met.1.** *The Löwenheim–Skolem Theorem fails for second-order logic: There are sentences with infinite models but no enumerable models.*

Proof. Recall that

$$\mathrm{Count} \equiv \exists z\, \exists u\, \forall X\, ((X(z) \land \forall x\, (X(x) \to X(u(x)))) \to \forall x\, X(x))$$

is true in a structure $\mathfrak{M}$ iff $|\mathfrak{M}|$ is enumerable, so $\lnot\mathrm{Count}$ is true in $\mathfrak{M}$ iff $|\mathfrak{M}|$ is non-enumerable. There are such structures—take any non-enumerable set as the domain, e.g., $\wp(\mathbb{N})$ or $\mathbb{R}$. So $\lnot\mathrm{Count}$ has infinite models but no enumerable models. □

Theorem met.2. *There are sentences with denumerable but no non-enumerable models.*

Proof. $\mathrm{Count} \land \mathrm{Inf}$ is true in $\mathbb{N}$ but not in any structure $\mathfrak{M}$ with $|\mathfrak{M}|$ non-enumerable. □

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Bibliography