

met.1 Introduction

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sec First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a **derivability** relation \vdash with the property that for any set of **sentences** Γ and **sentence** Q , $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$. This means in particular that the validities of first-order logic are **computably enumerable**. There is a computable function $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L})$ such that the values of f are all and only the valid **sentences** of \mathcal{L} . This is so because **derivations** can be enumerated, and those that **derive** a single **sentence** are then mapped to that **sentence**. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we'll show that second-order logic is not only undecidable, but its validities are not even **computably enumerable**. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set Γ of **sentences** is satisfiable, Γ itself is satisfiable) and the Löwenheim–Skolem Theorem holds for it (if Γ has an infinite model it has a **denumerable** model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of **domains** that first-order logic cannot.

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Bibliography