met.1 Second-order Logic is not Compact

sol:met:com:sec Call a set of sentences Γ *finitely satisfiable* if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences Γ is finitely satisfiable, it is satisfiable. This property is called *compactness*. It has an equivalent version involving entailment: if $\Gamma \vDash \varphi$, then already $\Gamma_0 \vDash \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \vDash \varphi$, by completeness $\Gamma \vdash \varphi$. But a derivation can only make use of finitely many sentences of Γ. explanation

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some φ without a finite subset entailing φ.

sol:met:com:thm:sol-undecidable **Theorem met.1.** *Second-order logic is not compact.*

Proof. Recall that

$$\mathrm{Inf} \equiv \exists u\,(\forall x\,\forall y\,(u(x) = u(y) \to x = y) \land \exists y\,\forall x\, y \neq u(x))$$

is satisfied in a structure iff its domain is infinite. Let $\varphi^{\geq n}$ be a sentence that asserts that the domain has at least n elements, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \ldots \exists x_n\,(x_1 \neq x_2 \land x_1 \neq x_3 \land \cdots \land x_{n-1} \neq x_n).$$

Consider the set of sentences

$$\Gamma = \{\neg\mathrm{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \ldots\}.$$

It is finitely satisfiable, since for any finite subset $\Gamma_0 \subseteq \Gamma$ there is some k so that $\varphi^{\geq k} \in \Gamma$ but no $\varphi^{\geq n} \in \Gamma$ for $n > k$. If $|\mathfrak{M}|$ has k elements, $\mathfrak{M} \vDash \Gamma_0$. But, Γ is not satisfiable: if $\mathfrak{M} \vDash \neg\mathrm{Inf}$, $|\mathfrak{M}|$ must be finite, say, of size k. Then $\mathfrak{M} \nvDash \varphi^{\geq k+1}$. □

Problem met.1. Give an example of a set Γ and a sentence φ so that $\Gamma \vDash \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \nvDash \varphi$.

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Bibliography