

syn.1 Valuations and Satisfaction

pl:syn:val:
sec

Definition syn.1 (Valuations). Let $\{\mathbb{T}, \mathbb{F}\}$ be the set of the two truth values, “true” and “false.” A *valuation* for \mathcal{L}_0 is a function \mathbf{v} assigning either \mathbb{T} or \mathbb{F} to the *propositional variables* of the language, i.e., $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{T}, \mathbb{F}\}$.

Definition syn.2. Given a *valuation* \mathbf{v} , define the evaluation function $\bar{\mathbf{v}}: \text{Frm}(\mathcal{L}_0) \rightarrow \{\mathbb{T}, \mathbb{F}\}$ inductively by:

$$\begin{aligned} \bar{\mathbf{v}}(\perp) &= \mathbb{F}; \\ \bar{\mathbf{v}}(\top) &= \mathbb{T}; \\ \bar{\mathbf{v}}(\rho_n) &= \mathbf{v}(\rho_n); \\ \bar{\mathbf{v}}(\neg\varphi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F}; \\ \mathbb{F} & \text{otherwise.} \end{cases} \\ \bar{\mathbf{v}}(\varphi \wedge \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \vee \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \rightarrow \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \leftrightarrow \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \bar{\mathbf{v}}(\psi); \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) \neq \bar{\mathbf{v}}(\psi). \end{cases} \end{aligned}$$

The clauses correspond to the following truth tables:

explanation

φ	$\neg\varphi$
\mathbb{T}	\mathbb{F}
\mathbb{F}	\mathbb{T}

φ	ψ	$\varphi \wedge \psi$
\mathbb{T}	\mathbb{T}	\mathbb{T}
\mathbb{T}	\mathbb{F}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{F}
\mathbb{F}	\mathbb{F}	\mathbb{F}

φ	ψ	$\varphi \vee \psi$
\mathbb{T}	\mathbb{T}	\mathbb{T}
\mathbb{T}	\mathbb{F}	\mathbb{T}
\mathbb{F}	\mathbb{T}	\mathbb{T}
\mathbb{F}	\mathbb{F}	\mathbb{F}

φ	ψ	$\varphi \rightarrow \psi$
\mathbb{T}	\mathbb{T}	\mathbb{T}
\mathbb{T}	\mathbb{F}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{T}
\mathbb{F}	\mathbb{F}	\mathbb{T}

φ	ψ	$\varphi \leftrightarrow \psi$
\mathbb{T}	\mathbb{T}	\mathbb{T}
\mathbb{T}	\mathbb{F}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{F}
\mathbb{F}	\mathbb{F}	\mathbb{T}

pl:syn:val:
thm:LocalDetermination

Theorem syn.3 (Local Determination). Suppose that \mathbf{v}_1 and \mathbf{v}_2 are *valuations* that agree on the propositional letters occurring in φ , i.e., $\mathbf{v}_1(\rho_n) = \mathbf{v}_2(\rho_n)$ whenever ρ_n occurs in some *formula* φ . Then $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$ also agree on φ , i.e., $\bar{\mathbf{v}}_1(\varphi) = \bar{\mathbf{v}}_2(\varphi)$.

Proof. By induction on φ . □

Definition syn.4 (Satisfaction). Using the evaluation function, we can define the notion of *satisfaction of a formula φ by a valuation \mathbf{v}* , $\mathbf{v} \models \varphi$, inductively as follows. (We write $\mathbf{v} \not\models \varphi$ to mean “not $\mathbf{v} \models \varphi$.”) pl:syn:val:
defn:satisfaction

1. $\varphi \equiv \perp$: $\mathbf{v} \not\models \varphi$.
2. $\varphi \equiv \top$: $\mathbf{v} \models \varphi$.
3. $\varphi \equiv p_i$: $\mathbf{v} \models \varphi$ iff $\mathbf{v}(p_i) = \mathbb{T}$.
4. $\varphi \equiv \neg\psi$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \not\models \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \models \psi$ and $\mathbf{v} \models \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \models \psi$ or $\mathbf{v} \models \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \not\models \psi$ or $\mathbf{v} \models \chi$ (or both).
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathbf{v} \models \varphi$ iff either both $\mathbf{v} \models \psi$ and $\mathbf{v} \models \chi$, or neither $\mathbf{v} \models \psi$ nor $\mathbf{v} \models \chi$.

If Γ is a set of formulas, $\mathbf{v} \models \Gamma$ iff $\mathbf{v} \models \varphi$ for every $\varphi \in \Gamma$.

Proposition syn.5. $\mathbf{v} \models \varphi$ iff $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$.

pl:syn:val:
prop:sat-value

Proof. By induction on φ . □

Problem syn.1. Prove [Proposition syn.5](#)

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Bibliography