## syn.1 Valuations and Satisfaction

pl:syn:val:sec

**Definition syn.1 (Valuations).** Let $\{\mathbb{T}, \mathbb{F}\}$ be the set of the two truth values, "true" and "false." A *valuation* for $\mathcal{L}_0$ is a function $\mathfrak{v}$ assigning either $\mathbb{T}$ or $\mathbb{F}$ to the propositional variables of the language, i.e., $\mathfrak{v} \colon \mathrm{At}_0 \to \{\mathbb{T}, \mathbb{F}\}$.

**Definition syn.2.** Given a valuation $\mathfrak{v}$, define the evaluation function $\overline{\mathfrak{v}} \colon \mathrm{Frm}(\mathcal{L}_0) \to \{\mathbb{T}, \mathbb{F}\}$ inductively by:

$$\begin{aligned}
\overline{\mathfrak{v}}(\bot) &= \mathbb{F};\\
\overline{\mathfrak{v}}(\top) &= \mathbb{T};\\
\overline{\mathfrak{v}}(p_n) &= \mathfrak{v}(p_n);\\
\overline{\mathfrak{v}}(\lnot\varphi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F};\\ \mathbb{F} & \text{otherwise.}\end{cases}\\
\overline{\mathfrak{v}}(\varphi \land \psi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{T};\\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{F}.\end{cases}\\
\overline{\mathfrak{v}}(\varphi \lor \psi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{T};\\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{F}.\end{cases}\\
\overline{\mathfrak{v}}(\varphi \to \psi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{T};\\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{F}.\end{cases}\\
\overline{\mathfrak{v}}(\varphi \leftrightarrow \psi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \overline{\mathfrak{v}}(\psi);\\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) \neq \overline{\mathfrak{v}}(\psi).\end{cases}
\end{aligned}$$

The clauses correspond to the following truth tables: explanation

| $\varphi$ | $\lnot\varphi$ |
|---|---|
| $\mathbb{T}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{T}$ |

| $\varphi$ | $\psi$ | $\varphi \land \psi$ |
|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{F}$ |

| $\varphi$ | $\psi$ | $\varphi \lor \psi$ |
|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ |
| $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{F}$ |

| $\varphi$ | $\psi$ | $\varphi \to \psi$ |
|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{T}$ |

| $\varphi$ | $\psi$ | $\varphi \leftrightarrow \psi$ |
|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{T}$ |

**Problem syn.1.** Consider adding to $\mathcal{L}_0$ a ternary connective $\diamondsuit$ with evaluation given by

$$\overline{\mathfrak{v}}(\diamondsuit(\varphi, \psi, \chi)) = \begin{cases} \overline{\mathfrak{v}}(\psi) & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T};\\ \overline{\mathfrak{v}}(\chi) & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F}.\end{cases}$$

 

Write down the truth table for this connective.

**Theorem syn.3 (Local Determination).** *Suppose that $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are valuations that agree on the propositional variables occurring in $\varphi$, i.e., $\mathfrak{v}_1(p_n) = \mathfrak{v}_2(p_n)$ whenever $p_n$ occurs in some formula $\varphi$. Then $\overline{\mathfrak{v}_1}$ and $\overline{\mathfrak{v}_2}$ also agree on $\varphi$, i.e., $\overline{\mathfrak{v}_1}(\varphi) = \overline{\mathfrak{v}_2}(\varphi)$.* pl:syn:val: thm:LocalDetermination

*Proof.* By induction on $\varphi$. □

**Definition syn.4 (Satisfaction).** We can inductively define the notion of *satisfaction of a formula $\varphi$ by a valuation* $\mathfrak{v}$, $\mathfrak{v} \vDash \varphi$, as follows. (We write $\mathfrak{v} \nvDash \varphi$ to mean "not $\mathfrak{v} \vDash \varphi$.") pl:syn:val: defn:satisfaction

1. $\varphi \equiv \bot$: $\mathfrak{v} \nvDash \varphi$.
2. $\varphi \equiv \top$: $\mathfrak{v} \vDash \varphi$.
3. $\varphi \equiv p_i$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v}(p_i) = \mathbb{T}$.
4. $\varphi \equiv \neg\psi$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \nvDash \psi$.
5. $\varphi \equiv (\psi \land \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \vDash \psi$ and $\mathfrak{v} \vDash \chi$.
6. $\varphi \equiv (\psi \lor \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \vDash \psi$ or $\mathfrak{v} \vDash \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \nvDash \psi$ or $\mathfrak{v} \vDash \chi$ (or both).
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{v} \vDash \varphi$ iff either both $\mathfrak{v} \vDash \psi$ and $\mathfrak{v} \vDash \chi$, or neither $\mathfrak{v} \vDash \psi$ nor $\mathfrak{v} \vDash \chi$.

If $\Gamma$ is a set of formulas, $\mathfrak{v} \vDash \Gamma$ iff $\mathfrak{v} \vDash \varphi$ for every $\varphi \in \Gamma$.

**Proposition syn.5.** $\mathfrak{v} \vDash \varphi$ *iff* $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$. pl:syn:val: prop:sat-value

*Proof.* By induction on $\varphi$. □

**Problem syn.2.** Prove Proposition syn.5

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## Bibliography