syn.1 Semantic Notions

pl:syn:sem: sec

We define the following semantic notions:

Definition syn.1.

1. A formula φ is *satisfiable* if for some $\mathfrak{v}$, $\mathfrak{v} \vDash \varphi$; it is *unsatisfiable* if for no $\mathfrak{v}$, $\mathfrak{v} \vDash \varphi$;
2. A formula φ is a *tautology* if $\mathfrak{v} \vDash \varphi$ for all valuations $\mathfrak{v}$;
3. A formula φ is *contingent* if it is satisfiable but not a tautology;
4. If Γ is a set of formulas, $\Gamma \vDash \varphi$ ("Γ entails φ") if and only if $\mathfrak{v} \vDash \varphi$ for every valuation $\mathfrak{v}$ for which $\mathfrak{v} \vDash \Gamma$.
5. If Γ is a set of formulas, Γ is *satisfiable* if there is a valuation $\mathfrak{v}$ for which $\mathfrak{v} \vDash \Gamma$, and Γ is *unsatisfiable* otherwise.

Problem syn.1. For each of the following four formulas determine whether it is (a) satisfiable, (b) tautology, and (c) contingent.

1. $(p_0 \to (\neg p_1 \to \neg p_0))$.
2. $((p_0 \land \neg p_1) \to (\neg p_0 \land p_2)) \leftrightarrow ((p_2 \to p_0) \to (p_0 \to p_1))$.
3. $(p_0 \leftrightarrow p_1) \to (p_2 \leftrightarrow \neg p_1)$.
4. $((p_0 \leftrightarrow (\neg p_1 \land p_2)) \lor (p_2 \to (p_0 \leftrightarrow p_1)))$.

pl:syn:sem: prop:semanticalfacts

Proposition syn.2.

1. *φ is a tautology if and only if $\emptyset \vDash \varphi$;*
2. *If $\Gamma \vDash \varphi$ and $\Gamma \vDash \varphi \to \psi$ then $\Gamma \vDash \psi$;*
3. *If Γ is satisfiable then every finite subset of Γ is also satisfiable;*
4. *pl:syn:sem: def:monotonicity* *Monotonicity: if $\Gamma \subseteq \Delta$ and $\Gamma \vDash \varphi$ then also $\Delta \vDash \varphi$;*
5. *pl:syn:sem: def:Cut* *Transitivity: if $\Gamma \vDash \varphi$ and $\Delta \cup \{\varphi\} \vDash \psi$ then $\Gamma \cup \Delta \vDash \psi$.*

Proof. Exercise. □

Problem syn.2. Prove Proposition syn.2

pl:syn:sem: prop:entails-unsat

Proposition syn.3. *$\Gamma \vDash \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.*

Proof. Exercise. □

Problem syn.3. Prove Proposition syn.3

pl:syn:sem: thm:sem-deduction

Theorem syn.4 (Semantic Deduction Theorem). *$\Gamma \vDash \varphi \to \psi$ if and only if $\Gamma \cup \{\varphi\} \vDash \psi$.*

Proof. Exercise. □

Problem syn.4. Prove Theorem syn.4

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Bibliography