## syn.1 Preliminaries

pl:syn:pre:sec

pl:syn:pre:thm:induction **Theorem syn.1 (*Principle of induction on formulas*).** *If some property $P$ holds for all the atomic formulas and is such that*

1. *it holds for $\neg\varphi$ whenever it holds for $\varphi$;*
2. *it holds for $(\varphi \wedge \psi)$ whenever it holds for $\varphi$ and $\psi$;*
3. *it holds for $(\varphi \vee \psi)$ whenever it holds for $\varphi$ and $\psi$;*
4. *it holds for $(\varphi \to \psi)$ whenever it holds for $\varphi$ and $\psi$;*
5. *it holds for $(\varphi \leftrightarrow \psi)$ whenever it holds for $\varphi$ and $\psi$;*

*then $P$ holds for all formulas.*

*Proof.* Let $S$ be the collection of all formulas with property $P$. Clearly $S \subseteq \mathrm{Frm}(\mathcal{L}_0)$. $S$ satisfies all the conditions of **??**: it contains all atomic formulas and is closed under the logical operators. $\mathrm{Frm}(\mathcal{L}_0)$ is the smallest such class, so $\mathrm{Frm}(\mathcal{L}_0) \subseteq S$. So $\mathrm{Frm}(\mathcal{L}_0) = S$, and every formula has property $P$. □

pl:syn:pre:prop:balanced **Proposition syn.2.** *Any formula in* $\mathrm{Frm}(\mathcal{L}_0)$ *is* balanced, *in that it has as many left parentheses as right ones.*

**Problem syn.1.** Prove Proposition syn.2

pl:syn:pre:prop:noinit **Proposition syn.3.** *No proper initial segment of a formula is a formula.*

**Problem syn.2.** Prove Proposition syn.3

**Proposition syn.4 (Unique Readability).** *Any formula $\varphi$ in* $\mathrm{Frm}(\mathcal{L}_0)$ *has exactly one parsing as one of the following*

1. $\bot$.
2. $\top$.
3. $p_n$ *for some* $p_n \in \mathrm{At}_0$.
4. $\neg\psi$ *for some formula* $\psi$.
5. $(\psi \wedge \chi)$ *for some formulas* $\psi$ *and* $\chi$.
6. $(\psi \vee \chi)$ *for some formulas* $\psi$ *and* $\chi$.
7. $(\psi \to \chi)$ *for some formulas* $\psi$ *and* $\chi$.
8. $(\psi \leftrightarrow \chi)$ *for some formulas* $\psi$ *and* $\chi$.

*Moreover, this parsing is* unique.

*Proof.* By induction on $\varphi$. For instance, suppose that $\varphi$ has two distinct readings as $(\psi \to \chi)$ and $(\psi' \to \chi')$. Then $\psi$ and $\psi'$ must be the same (or else one would be a proper initial segment of the other); so if the two readings of $\varphi$ are distinct it must be because $\chi$ and $\chi'$ are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis. □

**Definition syn.5 (Uniform Substitution).** If $\varphi$ and $\psi$ are formulas, and $p_i$ is a propositional variable, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of $p_i$ by an occurrence of $\psi$ in $\varphi$; similarly, the simultaneous substitution of $p_1, \ldots, p_n$ by formulas $\psi_1, \ldots, \psi_n$ is denoted by $\varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$.

**Problem syn.3.** For each of the five formulas below determine whether the formula can be expressed as a substitution $\varphi[\psi/p_i]$ where $\varphi$ is (i) $p_0$; (ii) $(\neg p_0 \wedge p_1)$; and (iii) $((\neg p_0 \to p_1) \wedge p_2)$. In each case specify the relevant substitution.

1. $p_1$
2. $(\neg p_0 \wedge p_0)$
3. $((p_0 \vee p_1) \wedge p_2)$
4. $\neg((p_0 \to p_1) \wedge p_2)$
5. $((\neg(p_0 \to p_1) \to (p_0 \vee p_1)) \wedge \neg(p_0 \wedge p_1))$

**Problem syn.4.** Give a mathematically rigorous definition of $\varphi[\psi/p]$ by induction.

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## Bibliography