syn.1 Preliminaries

pl:syn:pre:sec

pl:syn:pre:thm:induction

Theorem syn.1 (*Principle of induction on formulas*). *If some property P holds for all the atomic formulas and is such that*

1. *it holds for $\neg\varphi$ whenever it holds for φ;*
2. *it holds for $(\varphi \wedge \psi)$ whenever it holds for φ and ψ;*
3. *it holds for $(\varphi \vee \psi)$ whenever it holds for φ and ψ;*
4. *it holds for $(\varphi \rightarrow \psi)$ whenever it holds for φ and ψ;*
5. *it holds for $(\varphi \leftrightarrow \psi)$ whenever it holds for φ and ψ;*

then P holds for all formulas.

Proof. Let S be the collection of all formulas with property P. Clearly $S \subseteq \mathrm{Frm}(\mathcal{L}_0)$. S satisfies all the conditions of **??**: it contains all atomic formulas and is closed under the logical operators. $\mathrm{Frm}(\mathcal{L}_0)$ is the smallest such class, so $\mathrm{Frm}(\mathcal{L}_0) \subseteq S$. So $\mathrm{Frm}(\mathcal{L}_0) = S$, and every formula has property P. □

pl:syn:pre:prop:balanced

Proposition syn.2. *Any formula in* $\mathrm{Frm}(\mathcal{L}_0)$ *is* balanced, *in that it has as many left parentheses as right ones.*

Problem syn.1. Prove Proposition syn.2

pl:syn:pre:prop:noinit

Proposition syn.3. *No proper initial segment of a formula is a formula.*

Problem syn.2. Prove Proposition syn.3

Proposition syn.4 (Unique Readability). *Any formula φ in* $\mathrm{Frm}(\mathcal{L}_0)$ *has exactly one parsing as one of the following*

1. $\bot$.
2. $\top$.
3. p_n *for some* $p_n \in \mathrm{At}_0$.
4. $\neg\psi$ *for some formula* ψ.
5. $(\psi \wedge \chi)$ *for some formulas* ψ *and* χ.
6. $(\psi \vee \chi)$ *for some formulas* ψ *and* χ.
7. $(\psi \rightarrow \chi)$ *for some formulas* ψ *and* χ.
8. $(\psi \leftrightarrow \chi)$ *for some formulas* ψ *and* χ.

Moreover, this parsing is unique.

Proof. By induction on φ. For instance, suppose that φ has two distinct readings as $(\psi \to \chi)$ and $(\psi' \to \chi')$. Then ψ and ψ' must be the same (or else one would be a proper initial segment of the other); so if the two readings of φ are distinct it must be because χ and χ' are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis. □

Definition syn.5 (Uniform Substitution). If φ and ψ are formulas, and p_i is a propositional variable, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of p_i by an occurrence of ψ in φ; similarly, the simultaneous substitution of $p_1, \ldots, p_n$ by formulas $\psi_1, \ldots, \psi_n$ is denoted by $\varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$.

Problem syn.3. For each of the five formulas below determine whether the formula can be expressed as a substitution $\varphi[\psi/p_i]$ where φ is (i) p_0; (ii) $(\neg p_0 \land p_1)$; and (iii) $((\neg p_0 \to p_1) \land p_2)$. In each case specify the relevant substitution.

1. p_1
2. $(\neg p_0 \land p_0)$
3. $((p_0 \lor p_1) \land p_2)$
4. $\neg((p_0 \to p_1) \land p_2)$
5. $((\neg(p_0 \to p_1) \to (p_0 \lor p_1)) \land \neg(p_0 \land p_1))$

Problem syn.4. Give a mathematically rigorous definition of $\varphi[\psi/p]$ by induction.

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Bibliography