syn.1 Propositional Formulas

pl:syn:fml:sec

Formulas of propositional logic are built up from *propositional variables*, the propositional constant $\bot$ and the propositional constant $\top$ using *logical connectives*.

1. A denumerable set At_0 of propositional variables p_0, p_1, ...
2. The propositional constant for falsity $\bot$.
3. The propositional constant for truth $\top$.
4. The logical connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditional), $\leftrightarrow$ (biconditional)
5. Punctuation marks: (,), and the comma.

We denote this language of propositional logic by $\mathcal{L}_0$.

intro

You may be familiar with different terminology and symbols than the ones we use above. Logic texts (and teachers) commonly use either $\sim$, $\neg$, and ! for "negation", $\wedge$, $\cdot$, and & for "conjunction". Commonly used symbols for the "conditional" or "implication" are $\rightarrow$, $\Rightarrow$, and $\supset$. Symbols for "biconditional," "bi-implication," or "(material) equivalence" are $\leftrightarrow$, $\Leftrightarrow$, and $\equiv$. The $\bot$ symbol is variously called "falsity," "falsum," "absurdity," or "bottom." The $\top$ symbol is variously called "truth," "verum," or "top."

pl:syn:fml:defn:formulas

Definition syn.1 (Formula). The set $\mathrm{Frm}(\mathcal{L}_0)$ of *formulas* of propositional logic is defined inductively as follows:

1. $\bot$ is an atomic formula.
2. $\top$ is an atomic formula.
3. Every propositional variable p_i is an atomic formula.
4. If φ is a formula, then $\neg\varphi$ is formula.
5. If φ and ψ are formulas, then $(\varphi \wedge \psi)$ is a formula.
6. If φ and ψ are formulas, then $(\varphi \vee \psi)$ is a formula.
7. If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.
8. If φ and ψ are formulas, then $(\varphi \leftrightarrow \psi)$ is a formula.
9. Nothing else is a formula.

explanation

The definition of formulas is an *inductive definition*. Essentially, we construct the set of formulas in infinitely many stages. In the initial stage, we pronounce all atomic formulas to be formulas; this corresponds to the first few cases of the definition, i.e., the cases for $\top$, $\bot$, p_i. "Atomic formula" thus means any formula of this form.

The other cases of the definition give rules for constructing new formulas out of formulas already constructed. At the second stage, we can use them to construct formulas out of atomic formulas. At the third stage, we construct new formulas from the atomic formulas and those obtained in the second stage, and so on. A formula is anything that is eventually constructed at such a stage, and nothing else.

Definition syn.2 (Syntactic identity). The symbol $\equiv$ expresses syntactic identity between strings of symbols, i.e., $\varphi \equiv \psi$ iff φ and ψ are strings of symbols of the same length and which contain the same symbol in each place.

The $\equiv$ symbol may be flanked by strings obtained by concatenation, e.g., $\varphi \equiv (\psi \lor \chi)$ means: the string of symbols φ is the same string as the one obtained by concatenating an opening parenthesis, the string ψ, the $\lor$ symbol, the string χ, and a closing parenthesis, in this order. If this is the case, then we know that the first symbol of φ is an opening parenthesis, φ contains ψ as a substring (starting at the second symbol), that substring is followed by $\lor$, etc.

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Bibliography