seq.1 Translating between G1c and G3c

pt:seq:tr:sec

We mentioned that both **G1c** and **G3c** are sound and complete for classical logic, i.e., they each prove all and only the sequents true in every structure. In particular, they prove the same sequents. We are now in a position to prove this fact without the detour through the soundness and completeness theorems, but using purely proof-theoretically. Such a proof provides *transformations* of derivations in **G1c** into derivations in **G3c** and vice versa.

Proposition seq.1. *If* $\mathbf{G1c} \vdash S$ *then* $\mathbf{G3c} \vdash S$.

Proof. We show that if π is a derivation in **G1c**, then there is a derivation π' of S in **G3c**. We do this by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, then π is an axiom. The axiom $\bot \Rightarrow$ is also an axiom in **G3c** (take the context Γ, Δ to be empty). In **G3c**, axioms $\varphi \Rightarrow \varphi$ with arbitrary, not just atomic φ are allowed. But we proved in **??** that all such sequents are derivable in **G3c**.

If $n > 0$, π ends in a rule R. The inductive hypothesis guarantees that there are derivations of the premises of that rule in **G3c**, i.e., that the immediate sub-derivations have translations in **G3c**. If the rule R is also a rule in **G3c**, we apply it to the translated derivations of the premises. For those rules that are no also rules of **G3c**, we have already shown that they are at least admissible. So, if rule R was $\wedge$L or $\vee$R, apply **??**. If it was a weakening rule, apply **??**. If it was a contraction rule, apply **??**. □

Proposition seq.2. *If* $\mathbf{G3c} \vdash S$ *then* $\mathbf{G1c} \vdash S$.

Proof. We again proceed by induction on the height of a derivation π of S. Every axiom of **G3c** can be derived in **G1c** from an axiom plus several applications of the WL and WR rules:

$$\dfrac{\dfrac{\chi \Rightarrow \chi}{\vdots\ \mathrm{WL}}}{\dfrac{\chi, \Gamma \Rightarrow \chi}{\vdots\ \mathrm{WR}}}\qquad \dfrac{\dfrac{\bot \Rightarrow}{\vdots\ \mathrm{WL}}}{\dfrac{\bot, \Gamma \Rightarrow}{\vdots\ \mathrm{WR}}}$$
$$\chi, \Gamma \Rightarrow \Delta, \chi \qquad \bot, \Gamma \Rightarrow \Delta$$

If π ends in a rule R, the inductive hypothesis yields derivations in **G1c** of the premises, and we can apply the same rule R to those derivations. The exceptions are the rules $\wedge$L and $\vee$R, which differ between **G1c** and **G3c**. The **G3c** versions are derivable in **G1c** as follows:

$$\dfrac{\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \psi, \Gamma \Rightarrow \Delta}\wedge\mathrm{L}}{\dfrac{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\mathrm{CL}}\wedge\mathrm{L} \qquad \dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi, \psi}\vee\mathrm{R}}{\dfrac{\Gamma \Rightarrow \Delta, , \varphi \vee \psi, \varphi \vee \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\mathrm{CR}}\vee\mathrm{R}$$

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Bibliography