Chapter udf

The Sequent Calculus

seq.1 Introduction

pt:seq:int:sec

Sequent calculi are derivation systems that were introduced in the 1930s by Gerhard Gentzen. They were intended less as a practical method for deriving things. Rather, Gentzen used them in order to be able to prove certain results about the possible structure of derivations and their property. These kinds of questions are exactly the questions with which proof theory is concerned.

Sequent calculi operate on *sequents*, as the name suggests, and not on formulas. A sequent is a pair of sequences or multisets of formulas. Gentzen's original systems (**LK** and **LJ**) used sequences; proof theorists now prefer to use multisets. A multiset is like a set, except that a multiset can contain an element more than once. Just as in sets, the order of elements does not matter, however. So the expressions $\{\varphi, \varphi, \psi\}$ and $\{\varphi, \psi, \varphi\}$ describe the same multiset, but that multiset is different from $\{\varphi, \psi\}$ and from $\{\varphi, \varphi, \psi, \psi\}$, since the latter contain different numbers of φ and ψ than the former.

Traditionally, multisets of formulas in proof theory are written using uppercase Greek letters. So when we write Γ, Δ, Π, Λ, or Θ, we intend these to be multisets of formulas of whatever logic we are working with. Traditionally also, instead of writing $\langle \Gamma, \Delta \rangle$ proof theorists separate the two components by a sequent symbol; we use $\Rightarrow$.

Definition seq.1 (Sequent). A *sequent* is an expression of the form

$$\Gamma \Rightarrow \Delta$$

where Γ and Δ are finite (possibly empty) multisets of formulas. Γ is called the *antecedent*, while Δ is the *succedent*.

Let γ be the conjunction of the formulas in Γ (taking the empty conjunction to be $\top$) and θ be the disjunction of the formulas in Δ (taking the empty disjunction to be $\bot$). Then the sequent $\Gamma \Rightarrow \Delta$ is true (in some structure $\mathfrak{M}$) iff $\gamma \to \theta$ is true. In classical logic, this amounts to the same as saying that either one of the formulas in Γ is false, or one of the formulas in Δ is true.

If Γ is a multiset of formulas, we write Γ, φ (or φ, Γ) for the result of adding φ to Γ. If Δ is also a multiset of formulas, then Γ, Δ is the multiset union of the two sequences—if Γ contains φ with *multiplicity* n (i.e., it contains n copies of φ) and Δ with multiplicity m, then Γ, Δ contains φ with multiplicity $n+m$. If the multiset on one side of a sequent is empty, we usually just leave a blank. For instance, $\Rightarrow \varphi$ is the sequent with an empty antecendent and the succedent contains φ with multiplicity 1.

A derivation in the sequent calculus is a tree of sequents, where the topmost (or initial) sequents are axioms in the system considered, and if a sequent stands below one or two other sequents, it must follow correctly by a rule of inference in the system. We will first consider two different sequent systems for classical logic, **G1c** and **G3c**. Their axioms and rules are given in Tables seq.1 and seq.2.

The systems are subtly different, and these differences mean that they have different proof theoretic properties. **G1c** has axioms $\varphi \Rightarrow \varphi$ with no other formulas allowed. **G3c** has axioms of the form $\chi, \Gamma \Rightarrow \Delta, \chi$ with arbitrary "side" formulas in Γ and Δ, but the formula χ appearing on both sides must be *atomic*. **G1c** has two versions each of the $\wedge$L and $\vee$R rules whereas **G3c** has just one each. And **G1c** has additional "structural rules" called "contraction" (CL, CR) and "weakening" (WL, WR).

The systems **G1c** and **G3c** happen to be sound and complete for classical logic. In other words, every sequent derivable in these systems is true in every structure, and every sequent that is true in every structure has a derivation. So the question *whether* a sequent has a derivation can also be answered by other methods of logic (e.g., those of model theory). And since both systems derive exactly the sequents true in every structure, they in particular prove the same sequents.

Proof theory, however, is not just concerned with *whether* something is provable, but *how*. Proof theorists consider questions such as "Can a certain rule always be avoided?," "Can this rule be added to the system without resulting in new sequents becoming provable?," or "Can derivations in one system be turned into derivations in another?" If the answer is "yes" to such a question, the proof of this fact is often proved by induction on the complexity of derivations, or, equivalently, by induction on the structure of derivations. This yields not just a proof of the fact (e.g., if **G1c** proves a sequent then **G3c** proves the same sequent), but a procedure (e.g., of turning derivations in **G1c** into derivations in **G3c**).

A central result in proof theory is the *cut-elimination theorem*. It shows that we can add the cut rule

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \text{Cut}$$

to the sequent systems without changing which sequents are derivable. Moreover, the proof provides a method that transforms any derivation using the rules of **G1c** (or **G3c**) together with Cut into one that only uses the rules of

Axioms

$$\varphi \Rightarrow \varphi \qquad\qquad \bot \Rightarrow$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{WL} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{WR}$$

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{CL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{CR}$$

Logical Rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\,\neg\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\,\neg\mathrm{R}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\,\wedge\mathrm{R}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\,\vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\,\to\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\,\to\mathrm{R}$$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\exists\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\,\exists\mathrm{R}$$

Table seq.1: Rules of **G1c**. In ∀R and ∃L, the constant symbol a must not occur in the conclusion sequent. Γ and Δ are multisets of formulas.

pt:seq:int:
tab:G1c

Axioms

$$\chi, \Gamma \Rightarrow \Delta, \chi \qquad\qquad \bot, \Gamma \Rightarrow \Delta$$

Logical Rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\,\neg\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\,\neg\mathrm{R}$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\,\wedge\mathrm{R}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\,\vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\,\to\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\,\to\mathrm{R}$$

$$\frac{\varphi(t), \forall x\, \varphi(x) \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\exists\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x), \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\,\exists\mathrm{R}$$

Table seq.2: Rules of **G3c**. In ∀R and ∃L, the constant symbol a must not occur in the conclusion sequent. Γ and Δ are multisets of formulas. In axioms, the formula χ must be atomic.

pt:seq:int:tab:G3c

G1c (or **G3c**). In other words, this procedure "eliminates" the Cut rule from derivations using it; hence the name.

seq.2 Rules and Derivations

We begin by giving some definitions and fixing some terminology. pt:seq:rp:sec

Consider, for example, the two rules in **G3c** that allow deriving sequents containing $\varphi \wedge \psi$ from sequents containing φ and ψ:

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\,\wedge\mathrm{R}$$

The sequents on top of the line are called the *premises* and the sequent below the line the *conclusion*. The formulas in Γ and Δ in each rule are called the *context* or the *side formulas*. The formula in the conclusion that isn't part of the context is called the *principal* formula, and the formulas in the premises not in the context the *active* (or *auxiliary*) formulas.

The quantifier rules are slightly more complicated. E.g., the rules in **G1c**

for $\forall$ are:

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

In the $\forall$L rule, the active formula is $\varphi(t)$ where t is a closed term, i.e., a term without variables. Such an inference is correct—that is, it matches the schematic rule $\forall$L—if there is some formula φ, some variable x, and some closed term t such that the principal formula in the conclusion is $\forall x\, \varphi$ and the active formula in the premise is $\varphi[t/x]$. The $\forall$R rule works the same, except that instead of any term t the active formula in the premise must be $\varphi[a/x]$, where a is a constant symbol. This constant is called the *eigenvariable* of the $\forall$R inference. The inference is only correct if the lower sequent does not contain a, that is, if a does not occur in Γ, Δ, or φ. This is called the *eigenvariable condition*.

Sequent systems like **G1c** and **G3c** consists of a collection of such schematically specified rules together with some (also schematically specified) sequents that count as axioms. Derivations in such a system are finite trees of sequents that are inductively generated from axioms using the rules of inference. Each leaf is an axiom, and every other sequent follows from the sequents immediately above it by one of the inference rules of the system.

pt:seq:rp: defn:proof

Definition seq.2. *Derivations* in a sequent system are finite trees of sequents inductively as follows:

pt:seq:rp: defn:prf:axiom

1. Every axiom is a derivation.

pt:seq:rp: defn:prf:one-prem

2. If π_1 is a derivation ending in a sequent S_1, and S is a sequent that follows from S_1 by a one-premise rule R, then

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ S_1\end{array}}{S}\,R$$

is a derivation.

pt:seq:rp: defn:prf:two-prem

3. If π_1 and π_2 are derivations ending in sequents S_1 and S_2, and S is a sequent that follows from S_1 andS_2 by a two-premise rule R, then

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ S_1\end{array} \qquad \begin{array}{c}\vdots\\ \pi_2\\ \vdots\\ S_2\end{array}}{S}\,R$$

is a derivation.

Definition seq.2 defines derivations as trees of sequents. We think of these trees as growing "upwards." The topmost (leaf) nodes of this tree are axioms

and called *initial sequents*. The bottom-most sequent is called the *end-sequent*. If π is a derivation with end-sequent S we also write $\pi \vdash S$. If there is a derivation of a sequent S we write $\vdash S$, possibly indicating the system in which the proof exists on the left of the $\vdash$ symbol, e.g., **G1c** $\vdash \varphi, \psi \Rightarrow \varphi \wedge \psi$. Every sequent in a derivation, with the exception of initial sequents, is the *conclusion* of an inference by some rule R. The one or two sequents standing immediately above a sequent in a derivation (i.e., the parents of the node) are called the *premises* of the inference.

The derivations used in the inductive construction of a derivation are called its *sub-derivations*. The sub-derivations ending in the premises of an inference are called the *immediate sub-derivations* of the inference.

It is often necessary to measure the complexity of derivations as a natural number. We could simply count the number of sequents in a derivation, but often a more useful measure is the height of a derivation: it is the maximal number of sequents between the end-sequent and an initial sequent. We can also give an inductive definition.

Definition seq.3. The *height* $\mathrm{ht}(\pi)$ of a derivation π is inductively defined as follows.

1. If π consists of a single axiom, $\mathrm{ht}(\pi) = 0$.
2. If π ends in an inference with one premise with immediate sub-derivation π_1, then $\mathrm{ht}(\pi) = \mathrm{ht}(\pi_1) + 1$.
3. If π ends in an inference with two premises with immediate sub-derivations π_1 and π_2, then $\mathrm{ht}(\pi) = \max(\mathrm{ht}(\pi_1), \mathrm{ht}(\pi_2)) + 1$.

If a sequent S has a derivation of height $\leq n$, we write $\vdash_n S$.

Example seq.4. In **G3c**, any sequent of the form $\chi, \Gamma \Rightarrow \Delta, \chi$ is an axiom, where χ must be atomic. Suppose θ and α are atomic sentences. Then $\theta, \alpha \Rightarrow \theta$ and $\theta, \alpha \Rightarrow \alpha$ are instances of the axiom $\chi, \Gamma \Rightarrow \Delta, \chi$. E.g., if we take $\chi \equiv \alpha$, $\Gamma = \{\theta\}$ and $\Delta = \emptyset$ in $\chi, \Gamma \Rightarrow \Delta, \chi$ we get exactly the sequent $\theta, \alpha \Rightarrow \alpha$. (Remember that we are dealing with multi*sets* and so $\theta, \alpha \Rightarrow \alpha$ and $\alpha, \theta \Rightarrow \theta$ are the same sequent.)

Since $\theta, \alpha \Rightarrow \theta$ is an instance of an axiom of **G3c**, it (by itself) counts as a derivation in **G3c**, and so does $\theta, \alpha \Rightarrow \alpha$ by Definition seq.2(1). Let's call them π_1 and π_2. By (3), we can take these two derivations and generate a new one (π_3) using the two-premise rule $\wedge$R:

$$\dfrac{\theta, \alpha \Rightarrow \theta \qquad \theta, \alpha \Rightarrow \alpha}{\theta, \alpha \Rightarrow \theta \wedge \alpha}\,\wedge\mathrm{R}$$

From π_3 in turn we obtain the derivation π_4

$$\pi_3\left\{\dfrac{\dfrac{\theta, \alpha \Rightarrow \theta \qquad \theta, \alpha \Rightarrow \alpha}{\theta, \alpha \Rightarrow \theta \wedge \alpha}\,\wedge\mathrm{R}}{\theta \wedge \alpha \Rightarrow \theta \wedge \alpha}\,\wedge\mathrm{L}\right\}\pi_4$$

using the one-premise rule $\wedge$L (using Definition seq.2(2)). The immediate sub-derivation of the last inference in π_4 is π_3. The immediate sub-derivations of the $\wedge$R inference are π_1 and π_2. The sub-derivations of π_4 are all of π_1, π_2, π_3, and π_4 itself. We have $\mathrm{ht}(\pi_1) = \mathrm{ht}(\pi_2) = 0$, $\mathrm{ht}(\pi_3) = 1$ and $\mathrm{ht}(\pi_4) = 2$. We have shown that $\mathbf{G3c} \vdash_4 \theta \wedge \alpha \Rightarrow \theta \wedge \alpha$ for any atomic θ and α.

seq.3 Examples of Proofs

pt:seq:ex:sec

To give derivations of sequents it is typically convenient to work your way upwards from the end-sequent by selecting a formula in it as the active formula and writing the premises of the corresponding rule on top of the sequent, then repeating the process until it leads to axioms. For instance, suppose we want to prove the sequent

$$(\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)$$

Consider $\chi \to (\theta \to \alpha)$: it is of the form $\varphi \to \psi$ and occurs on the right of the sequent. Matching the entire sequent with the $\to$R rule of **G1c**,

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} \to\mathrm{R}$$

suggests we should take $\Gamma = \{\chi \to (\theta \to \alpha)\}$, $\Delta = \emptyset$, $\varphi \equiv \chi$ and $\psi \equiv \theta \to \alpha$. So the last inference of a proof could be

$$\frac{\chi, (\chi \wedge \theta) \to \alpha \Rightarrow \theta \to \alpha}{(\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)} \to\mathrm{R}$$

If we repeat this idea, focusing on $\theta \to \alpha$ as active formula, we generate

$$\dfrac{\dfrac{\theta, \chi, (\chi \wedge \theta) \to \alpha \Rightarrow \alpha}{\chi, (\chi \wedge \theta) \to \alpha \Rightarrow \theta \to \alpha} \to\mathrm{R}}{(\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)} \to\mathrm{R}$$

Now we have to focus on $(\chi \wedge \theta) \to \alpha$ and use the $\to$L rule of **G1c**,

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta} \to\mathrm{L}$$

This leads us to

$$\dfrac{\dfrac{\dfrac{\theta, \chi \Rightarrow \alpha, \chi \wedge \theta \qquad \theta, \chi, \alpha \Rightarrow \alpha}{\theta, \chi, (\chi \wedge \theta) \to \alpha \Rightarrow \alpha} \to\mathrm{L}}{\chi, (\chi \wedge \theta) \to \alpha \Rightarrow \theta \to \alpha} \to\mathrm{R}}{(\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)} \to\mathrm{R}$$

Using the formula $\chi \wedge \theta$ in the left topmost sequent as principal and the $\wedge$R rule gives us

$$\dfrac{\dfrac{\dfrac{\theta, \chi \Rightarrow \alpha, \chi \qquad \theta, \chi \Rightarrow \alpha, \theta}{\theta, \chi \Rightarrow \alpha, \chi \wedge \theta}\wedge\mathrm{R} \qquad \theta, \chi, \alpha \Rightarrow \alpha}{\theta, \chi, (\chi \wedge \theta) \to \alpha \Rightarrow \alpha}\to\mathrm{R}}{\chi, (\chi \wedge \theta) \to \alpha \Rightarrow \theta \to \alpha}\to\mathrm{R}}{(\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)}\to\mathrm{R}$$

So far so good. Note that everything we did also works in **G3c**, since the rules we've made use of so far are the same in **G1c** and **G3c**. We've arrived at a derivation fragment where all top sequents contain the same formula on the left as on the right. But they are not quite axioms yet.

In **G1c**, we have to give derivations of the topmost sequents from axioms of the form $\varphi \Rightarrow \varphi$. This is easily done using the weakening rules, e.g., the leftmost topmost sequent $\theta, \chi \Rightarrow \alpha, \chi$ can be derived as follows:

Overall, we have a derivation in **G1c** which looks like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\mathrm{WL}}{\theta, \chi \Rightarrow \alpha, \chi}\mathrm{WR} \qquad \dfrac{\dfrac{\theta \Rightarrow \theta}{\theta, \chi \Rightarrow \theta}\mathrm{WL}}{\theta, \chi \Rightarrow \alpha, \theta}\mathrm{WR}}{\theta, \chi \Rightarrow \alpha, \chi \wedge \theta}\wedge\mathrm{R} \qquad \dfrac{\dfrac{\alpha \Rightarrow \alpha}{\chi, \alpha \Rightarrow \chi}\mathrm{WL}}{\theta, \chi, \alpha \Rightarrow \alpha}\mathrm{WL}}{\theta, \chi, (\chi \wedge \theta) \to \alpha \Rightarrow \alpha}\to\mathrm{L}}{\chi, (\chi \wedge \theta) \to \alpha \Rightarrow \theta \to \alpha}\to\mathrm{R}}{(\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)}\to\mathrm{R}$$

The structure of this proof (in particular, its height) is independent of what the formulas χ, θ, and α are. We can replace these symbols in the derivation above by any formulas and the result will still be a derivation in **G1c**. The proof in **G1c** is *schematic*.

In **G3c**, axioms are sequents of the form $\varphi, \Gamma \Rightarrow \Delta, \varphi$ where φ must be atomic. So although $\theta, \chi \Rightarrow \alpha, \chi$ looks like an axiom of **G3c** ($\Gamma = \{\theta\}$; $\Delta = \{\alpha\}$), it is only *really* an axiom if χ in fact is atomic. Our proof fragment is only a complete proof in **G3c** if χ, θ, and α are atomic formulas.

Although we haven't, strictly speaking, shown that

$$\mathbf{G1c} \vdash (\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha),$$

in practice this is all we need to do (and can do). That's because every sequent of the form $\varphi, \Gamma \Rightarrow \Delta, \varphi$ is provable in **G3c**, even if φ itself is not atomic. We can show this by induction on the depth $\mathrm{dp}(\varphi)$ of φ.

pt:seq:ex:
defn:depth

Definition seq.5. The *depth* $\mathrm{dp}(\varphi)$ of a formula is defined as follows:

1. If φ is atomic, $\mathrm{dp}(\varphi) = 0$.

2. If $\varphi \equiv \neg\psi$, $\varphi \equiv \forall x\, A$, or $\varphi \equiv \exists x\, A$, then $\mathrm{dp}(\varphi) = \mathrm{dp}(\psi) + 1$.

3. If $\varphi \equiv (\psi \wedge \chi)$, $(\psi \vee \chi)$, or $(\psi \to \chi)$, then $\mathrm{dp}(\varphi) = \max(\mathrm{dp}(\psi), \mathrm{dp}(\chi)) + 1$.

It is not hard to prove that $\mathrm{dp}(A(x)) = \mathrm{dp}(A(t))$ for any term t. The important fact for us is that in any rule, the depth of the principal formula is always greater than the depth of the active formulas. For instance, we have:

$$\begin{aligned} \mathrm{dp}(P(x)) = \mathrm{dp}(P(a)) &= 0, \\ \mathrm{dp}(\forall x\, P(x)) &= 1, \\ \mathrm{dp}((\forall x\, P(x) \to P(a))) &= \max(1, 0) + 1 = 2. \end{aligned}$$

We can use this to prove some results by induction on $\mathrm{dp}(\varphi)$, such as the following.

pt:seq:ex:prop:G1c-axioms

Proposition seq.6. *For any φ, there is a derivation in* **G1c** *of $\varphi \Rightarrow \varphi$ in which all axioms are atomic.*

Proof. By induction on $\mathrm{dp}(\varphi)$. If $\mathrm{dp}(\varphi) = 0$ then φ is atomic and $\varphi \Rightarrow \varphi$ already is a derivation in **G1c** of the required form. If $\mathrm{dp}(\varphi) > 0$, φ is constructed from formulas of lower depth, e.g, $\varphi \equiv \neg\psi$, $\varphi \equiv (\psi \wedge \chi)$, or $\varphi \equiv \forall x\, A(x)$. The inductive hypothesis is that for all θ with $\mathrm{dp}(\theta) < \mathrm{dp}(\varphi)$, $\mathbf{G1c} \vdash \theta \Rightarrow \theta$ from atomic axioms.

If $\varphi \equiv \neg\psi$, then $\mathrm{dp}(\psi) < \mathrm{dp}(\varphi)$ and by inductive hypothesis, there is a derivation π_1 in **G1c** from atomic axioms of $\psi \Rightarrow \psi$. We can extend this to a derivation of $\neg\psi \Rightarrow \neg\psi$ as follows:

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{\psi \Rightarrow \psi}}{\neg\psi, \psi \Rightarrow}\,\neg\mathrm{L}}{\neg\psi \Rightarrow \neg\psi}\,\neg\mathrm{R}$$

If $\varphi \equiv (\psi \wedge \chi)$, then $\mathrm{dp}(\psi) < \mathrm{dp}(\varphi)$ and $\mathrm{dp}(\chi) < \mathrm{dp}(\varphi)$. By inductive hypothesis, there are derivations π_1 and π_2 in **G1c** from atomic axioms of $\psi \Rightarrow \psi$ and $\chi \Rightarrow \chi$. We can extend these to a derivation of $\psi \wedge \chi \Rightarrow \psi \wedge \chi$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{\psi, \Rightarrow \psi}}{\psi, \chi, \Rightarrow \psi}\,\mathrm{WL}}{\psi \wedge \chi, \chi \Rightarrow \psi}\,\wedge\mathrm{L} \qquad \dfrac{\dfrac{\dfrac{\vdots\,\pi_2}{\chi, \Rightarrow \chi}}{\psi, \chi, \Rightarrow \chi}\,\mathrm{WL}}{\psi \wedge \chi, \chi \Rightarrow \chi}\,\wedge\mathrm{L}}{\psi \wedge \chi, \psi \wedge \chi \Rightarrow \chi}\,\wedge\mathrm{L}}{\psi \wedge \chi \Rightarrow \psi \wedge \chi}\,\wedge\mathrm{R}}{\psi \wedge \chi \Rightarrow \psi}\,\mathrm{CL}$$

Now suppose $\varphi \equiv \forall x\, B(x)$. Take a constant symbol c that does not occur in $\forall x\, B(x)$. Then $\mathrm{dp}(B(c)) = \mathrm{dp}(\psi(x)) < \mathrm{dp}(\varphi)$ and by inductive hypothesis, there is a derivation π_1 in **G1c** from atomic axioms of $\psi(c) \Rightarrow \psi(c)$. We can extend this to a derivation of $\forall x\, B(x) \Rightarrow \forall x\, B(x)$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\psi(c) \Rightarrow \psi(c)}\,\forall\mathrm{L}}{\forall x\, \psi(x) \Rightarrow \psi(c)}\,\forall\mathrm{R}$$

The remaining cases are left as exercises. □

Problem seq.1. Complete the proof of Proposition seq.6 by carrying out the cases where $\varphi \equiv (\psi \vee \chi)$, $\varphi \equiv (\psi \to \chi)$ and $\varphi \equiv \exists x\, B(x)$.

For our purposes, we could also have used the derivation

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\psi \Rightarrow \psi}}{\Rightarrow \psi, \neg\psi}\,\neg\mathrm{R}}{\neg\psi \Rightarrow \neg\psi}\,\neg\mathrm{L}$$

in the case where $\varphi \equiv \neg\psi$. However, there are sequent systems in which this wouldn't have worked. The intuitionistic sequent calculus **G1i** is like **G1c**, except it restricts all sequents to at most one formula on the right side of a sequent. If $\Delta = \emptyset$, then the first derivation would also be a derivation in **G1i**, but the second would not be, since one sequent contains both ψ and $\neg\psi$ on the right.

Note that even in **G1c** we could not have used an alternate order of rules in the case where $\varphi \equiv \forall x\, \psi(x)$. The following is *not* a derivation

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\psi(c) \Rightarrow \psi(c)}}{\psi(c) \Rightarrow \forall x\, \psi(x)}\,\forall\mathrm{R}}{\forall x\, \psi(x) \Rightarrow \forall x\, \psi(x)}\,\forall\mathrm{L}$$

since the eigenvariable condition on $\forall\mathrm{R}$ is violated.

Proposition seq.7. *Any sequent of the form $\varphi, \Gamma \Rightarrow \Delta, \varphi$ (with φ not necessarily atomic) is provable in* **G3c**. *pt:seq:ex: prop:G3c-axioms*

Proof. The proof is very similar to the one of Proposition seq.6, but we must be careful with the inductive hypothesis. The case where $n = \mathrm{dp}(\varphi) = 0$ is again easy. If $n = 0$, φ is atomic, and $\varphi, \Gamma \Rightarrow \Delta, \varphi$ is an axiom of **G3c**.

Now suppose $n > 0$. Again we distinguish cases. The inductive hypothesis is: for all D with $\mathrm{dp}(\theta) < n$, and all Π and Λ, $\mathbf{G3c} \vdash \theta, \Pi \Rightarrow \Lambda, \theta$. Here is the case for $\varphi \equiv (\psi \wedge \chi)$. We use the inductive hypothesis twice: once for $\theta = \psi$, $\Pi = \chi, \Gamma$, and $\Lambda = \Delta$ to get a derivation π_1 of $\psi, \chi, \Gamma \Rightarrow \Delta, \psi$, and once for $\theta = \chi$, $\Pi = \psi, \Lambda$, and $\Lambda = \Delta$ to get a derivation π_2 of $\psi, \chi, \Gamma \Rightarrow \Delta, \chi$. Here is the derivation of $\psi \wedge \chi, \Gamma \Rightarrow \Delta, \psi \wedge \chi$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\end{array}}{\dfrac{\psi, \chi, \Gamma \Rightarrow \Delta, \psi}{\psi \wedge \chi, \Gamma \Rightarrow \Delta, \psi}\wedge\mathrm{L}} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_2\end{array}}{\psi, \chi, \Gamma \Rightarrow \Delta, \chi}}{\psi \wedge \chi, \Gamma \Rightarrow \Delta, \chi}\wedge\mathrm{L}}{\psi \wedge \chi, \Gamma \Rightarrow \Delta, \psi \wedge \chi}\wedge\mathrm{R}$$

For $\varphi \equiv \forall x\, \psi(x)$, pick a constant symbol not in $\forall x\, \psi(x)$, Γ, or Δ. Apply the inductive hypothesis for $\theta = \psi(c)$, $\Pi = \forall x\, \psi(x), \Gamma$ and $\Lambda = \Delta$. This gives a derivation π_1 of $\psi(c), \forall x\, \psi(x), \Gamma \Rightarrow \Delta, \psi(c)$, from which we obtain:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\end{array}}{\psi(c), \forall x\, \psi(x), \Gamma \Rightarrow \Delta, \psi(c)}}{\forall x\, \psi(x), \Gamma \Rightarrow \Delta, \psi(c)}\forall\mathrm{L}}{\forall x\, \psi(x), \Gamma \Rightarrow \Delta, \forall x\, \psi(x)}\forall\mathrm{R}$$

The eigenvariable condition on $\forall$R is satisfied by the way we chose c. □

Problem seq.2. Continue the proof of Proposition seq.7 by carrying out the cases where $\varphi \equiv (\psi \to \chi)$ and $\varphi \equiv \exists x\, B(x)$.

seq.4 Interpretation of Rules

pt:seq:irl:sec

Recall the interpretation of a sequent: A sequent $\Gamma \Rightarrow \Delta$ is true in a structure $\mathfrak{M}$ if either at least one $\varphi \in \Gamma$ is false or at least one $\varphi \in \Delta$ is true. We can read the logical rules of **G3c** as codifying the truth conditions for their principal formulas. Consider $\to$R. Its principal formula is $\varphi \to \psi$. It is true if either φ is false or ψ is true. Hence the sequent $\Rightarrow \varphi \to \psi$ is true iff $\varphi \Rightarrow \psi$ is true. If we add the context formulas Γ, Δ to the antecedents and succedents of these sequents, we get exactly the conclusion and premise of the $\to$R rule. The situation is similar for $\to$L. $\varphi \to \psi$ is false if φ is true and ψ is false. So $\varphi \to \psi \Rightarrow$ is true iff both $\Rightarrow \varphi$ and $\psi \Rightarrow$ are true. The conclusion of $\to$L is equivalent to the conjunction of its premises. This pattern ensures that the rules of **G3c** are sound (if all premises are true, the conclusion is true). They also ensure completeness.

In fact, the truth conditions for any truth-functional connective can be described in this way as a set of sequents. This follows from the fact that every truth-function in classical logic can be described by a formula in conjunctive normal form: a conjunction of disjunctions of atomic and negated atomic formulas. For instance, take $\varphi \oplus \psi$ to be true iff exactly one of φ and ψ is true, i.e., "exclusive or." Then $\varphi \oplus \psi$ is true iff $(\varphi \vee \psi) \wedge (\neg\varphi \vee \neg\psi)$ is, and it is false if $(\neg\varphi \vee \psi) \wedge (\varphi \vee \neg\psi)$ is true. Hence, we could introduce sequent calculus rules for $\oplus$ as

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta, \varphi}{\varphi \oplus \psi, \Gamma \Rightarrow \Delta} \oplus\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi \qquad \varphi, \psi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi \oplus \psi} \oplus\text{R}$$

and these rules would also be sound and complete. Their conclusions are true iff all premises are true.

Problem seq.3. Suppose $\varphi \downarrow \psi$ is true iff neither φ nor ψ is true ("NOR"). What would its **G3c** rules be? (Find two sequents which are true at the same time iff $\varphi \downarrow \psi$ is true. That gives you the $\downarrow$R rule. Then find one sequent that is true iff $\varphi \downarrow \psi$ is false. That gives you the $\downarrow$L rule.)

In **G3c**, each connective and quantifier has exactly two rules: a left rule and a right rule. In **G1c**, however, there are two $\wedge$L and two $\vee$R rules. The reason is that Gentzen's original system **LK**, on which **G1c** is modeled, had a variant **LJ** for an important non-classical logic, viz., intuitionistic logic. To get a system that was sound and complete for intuitionistic logic, the succedents of any sequent in **LJ** is restricted to contain at most one formula. This makes it impossible to use the $\vee$R rule of **G3c**, as its premise contains both φ and ψ in the succedent. Gentzen thus used a pair of $\vee$R rules for **LJ**, and used the same for **LK**. The intuitionistic version **G1i** similarly is just **G1c** with all succedents restricted to at most one formula. (**G3c** also has an intuitionistic version, but it some of its rules differ from the corresponding rules in **G3c**. For instance, it, too, uses a pair of $\vee$R rules.)

That the structural rules of **G1c** are sound is easy to see. For instance, if $\Gamma \Rightarrow \Delta$ contains a false formula on the right or a true formula on the left, so does $\Gamma \Rightarrow \Delta, \varphi$: the same one, whether φ is true or false is irrelevant. So if the premise of WR is true, so is its conclusion. Note that here the converse is not true. The conclusion may be true because φ is true, and then the premise is not guaranteed to be true.

Why have structural rules at all? Contraction (CR, CL) is needed for instance to derive $\Rightarrow \varphi \vee \neg\varphi$. If you start with $\varphi \Rightarrow \varphi$ you first get $\Rightarrow \varphi, \neg\varphi$ by $\neg$R. Then you can use the two versions of $\vee$R in succession, once to get $\varphi \vee \neg\varphi$ from $\neg\varphi$, and once to get $\varphi \vee \neg\varphi$ from φ. What you get is $\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi$. So to get $\Rightarrow \varphi \vee \neg\varphi$ in **G1c** you need to be able to

"contract" two occurrences of the same formula in a derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\,\neg\mathrm{R}}{\Rightarrow \varphi, \varphi \vee \neg\varphi}\,\vee\mathrm{R}}{\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi}\,\vee\mathrm{R}}{\Rightarrow \varphi \vee \neg\varphi}\,\mathrm{CR}$$

There are in fact valid sequents which you could not derive in **G1c** without weakening or contraction. For instance, a derivation of the sequent $\varphi \Rightarrow \psi \to \varphi$ in **G1c** has to end in $\to$R (since only $\psi \to \varphi$ can be principal), and that requires $\psi, \varphi \Rightarrow \varphi$ as a premise. This, in turn, requires WL to get it from the axiom $\varphi \Rightarrow \varphi$:

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\psi, \varphi \Rightarrow \varphi}\,\mathrm{WL}}{\varphi \Rightarrow \psi \to \varphi}\,{\to}\mathrm{R}$$

In **G3c** the contraction and weakening rules are not needed at all. Weakening is built into the axioms. Contraction can mostly be avoided by merging the two copies of the context Γ, Δ in rules with two premises into one. Both **G1c** and **G3c** have such "context sharing" rules. In one-premise rules, contraction is only needed for $\vee$R, $\wedge$L, $\exists$R, and $\forall$L. The versions of these rules in **G3c** make contraction entirely superfluous. There is a system **G2c** (see Table seq.3) in which the two-premise rules have independent contexts. In **G2c** contraction is even more important than in **G1c**.

seq.5 Regular Proofs and Substitution

pt:seq:qua:sec

The rules for the quantifiers introduce some complications owing to the "eigenvariable conditions" that apply to the $\exists$L and $\forall$R rules. Most of these complications can be dealt with by ensuring that the constant symbol a in these inferences is never reused. Let's introduce the following definition.

Definition seq.8. A derivation π in **G1c** or **G3c** is *regular* if

1. no eigenvariable a is used as the eigenvariable of more than one $\exists$L or $\forall$R inference, and
2. if a is the eigenvariable of a $\exists$L or $\forall$R inference, it only occurs above that inference in π.

pt:seq:qua:lem:subst-regular

Lemma seq.9. *If π is regular, ends in $\Gamma \Rightarrow \Delta$, c is a constant symbol not used as an eigenvariable in π, and t is a term not containing any eigenvariables of π, then $\pi[t/c]$ resulting from π by replacing every occurrence of c by t is a regular derivation. The end-sequent of $\pi[t/c]$ is $\Gamma[t/c] \Rightarrow \Delta[t/c]$, where $\Gamma[t/c] = \{\varphi(t) : \varphi(c) \in \Gamma\}$.*

Proof. By induction on $\mathrm{ht}(\pi)$. If $\mathrm{ht}(\pi) = 0$, it consists simply of an axiom. Then $\pi[t/c]$ is also an axiom, since any formula $\varphi(c)$ in it turns into $\varphi(t)$.

Axioms

$$\varphi \Rightarrow \varphi \qquad\qquad \bot \Rightarrow$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{WL} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{WR}$$

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{CL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{CR}$$

Logical Rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\,\neg\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\,\neg\mathrm{R}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \Gamma_2 \Rightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \wedge \psi}\,\wedge\mathrm{R}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L}$$

$$\frac{\varphi, \Gamma_1 \Rightarrow \Delta_1 \qquad \psi, \Gamma_2 \Rightarrow \Delta_2}{\varphi \vee \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,\vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \psi, \Gamma_2 \Rightarrow \Delta_2}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,\to\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\,\to\mathrm{R}$$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\exists\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\,\exists\mathrm{R}$$

Table seq.3: Rules of **G2c**. In ∀R and ∃L, the constant symbol a must not occur in the conclusion sequent. Γ and Δ are multisets of formulas.

pt:seq:irl:
tab:G2c

If $\mathrm{ht}(\pi) > 0$ we distinguish cases according to the last inference in π. The cases for structural rules (in **G1c**) and logical rules for propositional connectives are easy and left as exercises. We consider the cases where π ends in a quantifier inference.

1. If the last inference is $\forall$R, π is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \pi'\end{array}}{\Gamma(c) \Rightarrow \Delta'(c), \varphi(a,c)} \forall\mathrm{R}$$
$$\Gamma(c) \Rightarrow \Delta'(c), \forall x\, \varphi(x,c)$$

   By induction hypothesis, $\pi'[t/c]$ is a regular derivation of $\Gamma(t) \Rightarrow \Delta'(t), \varphi(a,t)$. Since a is an eigenvariable of π, a does not occur in t. Therefore, the last inference in $\pi[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\\ \pi'[t/c]\end{array}}{\Gamma(t) \Rightarrow \Delta'(t), \varphi(a,t)} \forall\mathrm{R}$$
$$\Gamma(t) \Rightarrow \Delta'(t), \forall x\, \varphi(x,t)$$

   is correct: the conclusion does not contain a.

2. If the last inference is $\forall$L in **G3c**, π is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \pi'\end{array}}{\varphi(s(c),c), \forall x\, \varphi(x,c), \Gamma(c) \Rightarrow \Delta'(c)} \forall\mathrm{R}$$
$$\forall x\, \varphi(x,c), \Gamma(c) \Rightarrow \Delta'(c)$$

   By induction hypothesis, $\pi'[t/c]$ is a regular derivation of $\varphi(s(t),t), \forall x\, \varphi(x,t), \Gamma(t) \Rightarrow \Delta'(t)$. The last inference in $\pi[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\\ \pi'[t/c]\end{array}}{\varphi(s(t),t), \forall x\, \varphi(x,t), \Gamma(t) \Rightarrow \Delta'(t)} \forall\mathrm{R}$$
$$\forall x\, \varphi(x,t), \Gamma(t) \Rightarrow \Delta'(t)$$

   is correct. The case for $\forall$L in **G1c** is similar, but slightly less complicated.

3. The last inference is $\exists$R or $\exists$L: exercise.

In each case, we added a sequent to the end of a regular derivation (or two regular derivations in the case of two-premise rules). The new sequent differs from the end-sequent of π only in that c was replaced by the term t. Moreover, the eigenvariables of π and Since t was assumed not to contain an eigenvariable of π and $\pi[t/c]$ are the same, the new end-sequent does not contain an eigenvariable of $\pi[t/c]$. Consequently, $\pi[t/c]$ is regular. □

Proposition seq.10. *If π is a derivation in* **G1c** *or* **G3c**, *then there is a regular derivation π' with the same structure (in particular, the same height) which differs from π only by replacing eigenvariables.*

Proof. For the purpose of the proof only, call an eigenvariable inference of π *clean* if its eigenvariable is not the eigenvariable of another inference nor does it occur anywhere in π except the immediate sub-derivation of the inference, and *dirty* otherwise. Let n be the number of dirty eigenvariable inferences in π. We show, by induction on n, that π can be transformed into the required regular derivation π'. If $n = 0$, all eigenvariable inferences in π are clean, so and π is regular, and there is nothing to prove. Otherwise, pick a highest eigenvariable inference, say $\forall$R. The sub-derivation π_1 of π ending in it is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \pi_2\\ \vdots\end{array}}{\Gamma \Rightarrow \Delta, \varphi(c)}\,\forall\mathrm{R}$$

Note that as c is an eigenvariable, it does not occur in Γ, Δ, or $\forall x\,\varphi(x)$. The proof π_2 is regular, since it contains no eigenvariable inferences at all. Let a be a constant symbol not occuring in π. By Lemma seq.9, $\pi_2[a/c]$ is a regular proof of $\Gamma \Rightarrow \Delta, \varphi(a)$, to which we can apply $\forall$R. If we replace π_1 in π by the proof π_1',

$$\dfrac{\begin{array}{c}\vdots\\ \pi_2[a/c]\\ \vdots\end{array}}{\Gamma \Rightarrow \Delta, \varphi(a)}\,\forall\mathrm{R}$$

we have replaced a dirty eigenvariable inference by a clean one, and the only difference is the replacement of the eigenvariable c by a. The resulting proof has $n - 1$ dirty eigenvariable inferences, and the result follows from the inductive hypothesis. □

We will not appeal to the proposition just proved explicitly, but we will assume that all derivations considered are regular—if they are not, we can always replace eigenvariables to make them regular.

seq.6 Admissible and Derivable Rules

pt:seq:adm:sec

Many results of proof theory concern—or use—results about whether everything that can be derived using a specific rule can also be derived without that rule. The cut-elimination theorem is the most well-known example. It shows that any sequent that can be derived using the rules of a sequent system with Cut can also be derived without Cut. Because this kind of property is so important, it has a name.

Definition seq.11. A rule R is *admissible* in a sequent system if any sequent that can be proved in that system together with the rule R can also be proved without R.

pt:seq:adm: ex:land-deriv

Example seq.12. The rule $\wedge$L takes different forms in **G1c** and **G3c**. In **G1c** there are two versions of the rule, one with the left conjunct φ of the principal formula $\varphi \wedge \psi$ in the premise, and one with the right conjunct ψ:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_1 \qquad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_2$$

By contrast, **G3c** has only one rule:

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_3$$

We might now ask, "Is **G3c**'s rule $\wedge\mathrm{L}_3$ admissible in **G1c**?" It is: we can directly simulate any inference using $\wedge\mathrm{L}_3$ using only the rules of **G1c**. In addition to $\wedge\mathrm{L}_1$ and $\wedge\mathrm{L}_2$ we'll also need the structural rule CL:

$$\dfrac{\dfrac{\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_1}{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_2}{\varphi \wedge \psi \Gamma \Rightarrow \Delta} \mathrm{CL}$$

Simulating a inferences using one rule using other rules is one way to show that a rule is admissible. But it is not the only way, and some rules are admissible but can't be so simulated. Rules that can be are called *derivable*.

Definition seq.13. A rule R,

$$\frac{S_1 \qquad \dots \qquad S_n}{S} R$$

is called *derivable* in a system if there is a schematic derivation of the sequent S using the rules and axioms of the system together with additional initial sequents of the forms of $S_1, \dots, S_n$.

The derivation given in Example seq.12 shows that $\wedge\mathrm{L}_3$ is a derivable rule in **G1c**, as that derivation is a schematic derivation of the conclusion of $\wedge\mathrm{L}_3$ from the premises of $\wedge\mathrm{L}_3$ as initial sequents using only rules of **G1c**. (It didn't use any of the axioms of **G1c**.)

pt:seq:adm: prop:lor-G3-adm

Proposition seq.14. *The rules* $\wedge$L *and* $\vee$R *of* **G3c** *are derivable in* **G1c**.

Proof. The proof for $\wedge$L is given in Example seq.12. The proof for $\vee$R is left as an exercise. □

Problem seq.4. Show that the rule $\vee$R of **G3c** is derivable in **G1c**.

Problem seq.5. Suppose $\leftrightarrow$ is a defined logical operator: $\varphi \leftrightarrow \psi$ is short for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. Show that the rules

1. $$\dfrac{\Gamma, \varphi, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \leftrightarrow \psi} \leftrightarrow\text{L}$$

2. $$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \leftrightarrow \psi} \leftrightarrow\text{R}$$

are derivable in (a) **G1c** and (b) **G3c**.

Example seq.15. The rule WR of **G1c**,

pt:seq:adm: ex:weak-adm

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,\text{WR}$$

is admissible in **G3c**. The idea of the proof is simple: suppose there is a derivation π of the premise $\Gamma \Rightarrow \Delta$ in **G3c**. Add the formula φ to the succedent of every sequent in π. Axioms remain axioms; correct inferences are still correct inferences. The end-formula of the new derivation is $\Gamma \Rightarrow \Delta, \varphi$.

The rule WR is, however, not derivable in **G3c**. For suppose it were, i.e., suppose one could find a derivation of $\Gamma \Rightarrow \Delta, \varphi$ using only the axioms and rules of **G3c** plus possibly the additional initial sequent $\Gamma \Rightarrow \Delta$. If this derivation is, as would be required for WR to be derivable, purely schematic, we could turn it into a derivation of $\Rightarrow \varphi$ by deleting Γ and Δ. This would turn the additional initial sequents $\Gamma \Rightarrow \Delta$ into the empty sequent $\Rightarrow$. Since the empty sequent contains no formulas at all, but all rules of **G3c** require at least one active formula in their premises, the empty sequent can't be a premise of any inference in this schematic derivation. Consequently, the derivation can only be schematic if it doesn't actually use the additional sequent $\Gamma \Rightarrow \Delta$ as an initial sequent. In other words, it would be a schematic derivation of $\Gamma \Rightarrow \Delta, \varphi$ using only the axioms and rules of **G3c** and we would have shown that any sequent of this form has a derivation in **G3c**. But this is impossible, since **G3c** is sound and so can only derive sequents that are true in every structure. Taking $\Gamma = \Delta = \emptyset$ and $\varphi \equiv \bot$, **G3c** would have to derive, e.g., the sequent $\Rightarrow \bot$, which it does not.

Let's give a nice inductive proof of the result that weakening is an admissible rule in **G3c**. In fact, a slightly stronger result holds, as the intuitive argument shows: by adding φ to the right side of every sequent of π, we do not change the structure of the derivation, in particular, we do not increase the height. Since this is important, it deserves its own definition.

Definition seq.16. A rule R,

$$\dfrac{S_1 \qquad \dots \qquad S_n}{S}\,R$$

is *height-preserving admissible* (or *hp-admissible*) in a system if, whenever $\vdash_{h_i} S_i$ for $i = 1, \dots, n$, then $\vdash_m S$ with $m = \max(h_1, \dots, h_n)$.

Note that it is enough for a rule to be height-preserving admissible that, whenever the premises S_i have derivations π_i with height $h_i = \mathrm{ht}(\pi_i)$, there is a derivation π of the conclusion of height $\mathrm{ht}(\pi)$ no greater than the maximum of the h_i. In particular, if there is only one premise S_1, it is enough that $\mathrm{ht}(\pi) \leq \mathrm{ht}(\pi_1)$. In the case of WR and WL we actually have $\mathrm{ht}(\pi) = \mathrm{ht}(\pi_1)$, but that is not necessary.

pt:seq:adm: prop:weak-G3c-adm

Proposition seq.17. *The rules* WR *and* WL *of* **G1c** *are height-preserving admissible in* **G3c**.

Proof. We have to show that if $\mathbf{G3c} \vdash \Gamma \Rightarrow \Delta$ with a derivation π of height $\mathrm{ht}(\pi) = n$ then there is a **G3c**-derivation π' of $\Gamma \Rightarrow \Delta, \varphi$ with $\mathrm{ht}(\pi') = n$. We prove this by induction on n. If $n = 0$, then π is just the axiom $\Gamma \Rightarrow \Delta$ (i.e., some atomic χ occurs in both Γ and Δ), and so is $\Gamma \Rightarrow \Delta, \varphi$. If $n = \mathrm{ht}(\pi) > 0$, the derivation π has a last inference. We distinguish cases according to which rule was used. We give just one example: suppose the last inference was $\wedge$R. Then $\Delta = \Delta', \psi \wedge \chi$ and the immediate sub-derivations π_1 and π_2 of the last inference have end-sequents $\Gamma \Rightarrow \Delta', \psi$ and $\Gamma \Rightarrow \Delta', \chi$, respectively. In other words, π looks like this:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta', \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \Gamma \Rightarrow \Delta', \chi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi}\,\wedge\mathrm{R}$$

We also have that $n = \max(\mathrm{ht}(\pi_1), \mathrm{ht}(\pi_2)) + 1$. So $\mathrm{ht}(\pi_1) < n$ and $\mathrm{ht}(\pi_2) < n$, and the inductive hypothesis applies: there are derivations π_1' and π_2' of $\Gamma \Rightarrow \Delta', \psi, \varphi$ and $\Gamma \Rightarrow \Delta', \chi, \varphi$. Using the $\wedge$R rule, we obtain a derivation π' of $\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi$:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1'\\ \Gamma \Rightarrow \Delta', \psi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2'\\ \Gamma \Rightarrow \Delta', \chi, \varphi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi}\,\wedge\mathrm{R}$$

We have $\mathrm{ht}(\pi') = \max(\mathrm{ht}(\pi_1'), \mathrm{ht}(\pi_2')) + 1 = \max(\mathrm{ht}(\pi_1), \mathrm{ht}(\pi_2)) + 1 = \mathrm{ht}(\pi)$.□

Proposition seq.17 is extremely useful. We'll introduce a notation for repeated application of it: if π is a derivation of $\Gamma \Rightarrow \Delta$ then $\pi[\Pi \Rightarrow \Lambda]$ is the result of applying admissibility of weakening on the left with all the formulas in Π and admissibility of weakening on the right with all formulas in Λ. It is a derivation of $\Gamma, \Pi \Rightarrow \Delta, \Lambda$.

seq.7 Invertibility of Rules

pt:seq:inv: sec

Definition seq.18. A rule R,

$$\frac{S_1 \qquad \dots \qquad S_n}{S}\, R$$

is *invertible* in a system if, whenever $\vdash S$ then $\vdash_{h_i} S_i$ for $i = 1, \dots, n$. It is *height-preserving invertible* (or *hp-invertible*) in a system if, whenever $\vdash_n S$ then $\vdash_n S_i$ for $i = 1, \dots, n$.

Lemma seq.19. *The rules of* **G3c** *for* $\neg$, $\wedge$, $\vee$, *and* $\to$ *are height-preserving invertible, that is:* pt:seq:inv: lem:G3c-invert

1. $\neg$R*: If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \neg\varphi$*, then* $\mathbf{G3c} \vdash_n \varphi, \Gamma \Rightarrow \Delta$.
2. $\neg$L*: If* $\mathbf{G3c} \vdash_n \Gamma, \neg\varphi \Rightarrow \Delta$*, then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi$.
3. $\wedge$R*: If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi \wedge \psi$*, then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \psi$.
4. $\wedge$L*: If* $\mathbf{G3c} \vdash_n \Gamma, \varphi \wedge \psi \Rightarrow \Delta$*, then* $\mathbf{G3c} \vdash_n \varphi, \psi, \Gamma \Rightarrow \Delta$.
5. $\vee$R*: If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi \vee \psi$*, then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi, \psi$.
6. $\vee$L*: If* $\mathbf{G3c} \vdash_n \varphi \vee \psi, \Gamma \Rightarrow \Delta$*, then* $\mathbf{G3c} \vdash_n \varphi, \Gamma \Rightarrow \Delta$ *and* $\mathbf{G3c} \vdash_n \psi, \Gamma \Rightarrow \Delta$.
7. $\to$R*: If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi \to \psi$*, then* $\mathbf{G3c} \vdash_n \varphi, \Gamma \Rightarrow \Delta, \psi$.
8. $\to$L*: If* $\mathbf{G3c} \vdash_n \varphi \to \psi, \Gamma \Rightarrow \Delta$*, then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathbf{G3c} \vdash_n \psi, \Gamma \Rightarrow \Delta$.

Proof. We have to show that, for any rule R of **G3c**, if there is a derivation π of the conclusion S of R, then there are also derivations π_i of the premises S_i of R, with $\mathrm{ht}(\pi_i) \leq \mathrm{ht}(\pi)$. Consider the rule $\wedge$L: suppose there is a derivation π with end-sequent of the form $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$. We have to show that there is a derivation π' of $\varphi, \psi, \Gamma \Rightarrow \Delta$ with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$. We do this by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, π consists of just an axiom: $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$. In **G3c**, axioms are of the form $\chi, \Gamma \Rightarrow \Delta, \chi$ with χ atomic. In other words, χ can't be $\varphi \wedge \psi$. If $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$ is an axiom, then some atomic χ must be an element of both Γ and Δ. But then $\varphi, \psi, \Gamma \Rightarrow \Delta$ is also an axiom, and we have found our derivation π' (which is also of height 0).

If $n > 0$ it ends in an inference. We have to prove the claim for n, but may assume that it is true for any derivation of height $< n$, even if the context is different. In other words, for any $k < n$ and any Π and Λ, if $\varphi \wedge \psi, \Pi \Rightarrow \Lambda$ has a derivation of height k, then there is a derivation of height $\leq k$ of $\varphi, \psi, \Pi \Rightarrow \Lambda$.

We have two cases: either $\varphi \wedge \psi$ on the left is principal, or it is not. If it is principal, the last inference is $\wedge$L and the immediate sub-derivation π_1 ends in $\varphi, \psi, \Gamma \Rightarrow \Delta$. The result holds in this case as $\mathrm{ht}(\pi_1) < \mathrm{ht}(\pi)$.

If $\varphi \wedge \psi$ is not principal, some other formula is and $\varphi \wedge \psi$ belongs to the context of the last inference. Then the inductive hypothesis applies to the immediate sub-derivations of the last rule R, which yields a derivation π'_1 or two derivations π'_1 and π'_2 of height no greater than the height(s) of the immediate sub-derivations of π. The end-sequents of π'_1 and π'_2 contain φ, ψ in the context on the left instead of $\varphi \wedge \psi$. Apply the rule R to obtain the proof π'. For instance, suppose π ends in $\to$L:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \varphi \wedge \psi, \Gamma' \Rightarrow \Delta, \chi & \varphi \wedge \psi, \Gamma', \theta \Rightarrow \Delta\end{array}}{\varphi \wedge \psi, \Gamma', \chi \to \theta \Rightarrow \Delta} \to\text{L}$$

Here $\chi \to \theta$ on the left is principal, and the context on the left is $\varphi \wedge \psi, \Gamma'$ (i.e., $\Gamma = \Gamma', \chi \to \theta$). The inductive hypothesis yields derivations π'_1 and π'_2 of $\varphi \wedge \psi, \Gamma' \Rightarrow \Delta, \chi$ and $\varphi \wedge \psi, \Gamma', \theta \Rightarrow \Delta$, respectively. These two sequents can again serve as premises for the $\to$L rule to get π':

$$\dfrac{\begin{array}{cc}\vdots\,\pi'_1 & \vdots\,\pi'_2\\ \varphi, \psi, \Gamma' \Rightarrow \Delta, \chi & \varphi, \psi, \Gamma', \theta \Rightarrow \Delta\end{array}}{\varphi, \psi, \Gamma', \chi \to \theta \Rightarrow \Delta} \to\text{L}$$

We have

$$\begin{aligned}\mathrm{ht}(\pi) &= \max(\mathrm{ht}(\pi_1), \mathrm{ht}(\pi_2)) + 1\\ \mathrm{ht}(\pi') &= \max(\mathrm{ht}(\pi'_1), \mathrm{ht}(\pi'_2)) + 1\end{aligned}$$

by definition, and

$$\begin{aligned}\mathrm{ht}(\pi'_1) &\leq \mathrm{ht}(\pi_1)\\ \mathrm{ht}(\pi'_2) &\leq \mathrm{ht}(\pi_2)\end{aligned}$$

by inductive hypothesis. So,

$$\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi).$$ □

Problem seq.6. Give the proof of Lemma seq.19 for $\to$R.

pt:seq:inv: lem:invert-quant

Lemma seq.20. *The rules* $\forall$R *and* $\exists$L *are height-preserving invertible in the following sense:*

1. *If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \forall x\, \varphi(x)$ *then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi(c)$, *and*
2. *if* $\mathbf{G3c} \vdash_n \exists x\, \varphi(x), \Gamma \Rightarrow \Delta$ *then* $\mathbf{G3c} \vdash_n \varphi(c), \Gamma \Rightarrow \Delta$,

for any c not occuring in Γ, Δ, and $\forall x\,\varphi(x)$ or $\exists x\,\varphi(x)$, respectively.

Proof. We show (1) and leave (2) as an exercise. The argument is similar to the proof of Lemma seq.19. In the inductive step, there are again two cases. The case where $\forall x\,\varphi(x)$ is principal again is quick: the premise of the last $\forall$R inference is of the required form $\Gamma \Rightarrow \Delta, \varphi(a)$ and the immediate subderivation π_1 ending in it has $\mathrm{ht}(\pi_1) < \mathrm{ht}(\pi)$. Moreover, the eigenvariable condition gives us that a does not occur in Γ, Δ, or $\forall x\,\varphi(x)$. Since we may assume that π_1 is regular, $\pi_1[c/a]$ is a derivation of the same height of $\Gamma \Rightarrow \Delta, \varphi(c)$ by Lemma seq.9.

In the second case, $\forall x\,\varphi(x)$ is not principal in the last inference; some other formula is. We again consider a case by way of example. Suppose the last inference is $\wedge$R. Some formula in Δ is of the form $\psi \wedge \chi$ and is principal. The derivation π ends in

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma \Rightarrow \Delta', \forall x\,\varphi(x), \psi & \Gamma \Rightarrow \Delta', \forall x\,\varphi(x), \chi\end{array}}{\Gamma \Rightarrow \Delta', \forall x\,\varphi(x), \psi \wedge \chi}\,\wedge\mathrm{R}$$

where $\Delta = \Delta', \psi \wedge \chi$. Let c be a constant symbol that does not appear in Δ; then clearly it also does not occur in Δ', ψ or in Δ', χ. So the inductive hypothesis applies to π_1 and π_2; we have derivations π_1' and π_2' with $\mathrm{ht}(\pi_1') \leq \mathrm{ht}(\pi_1)$ and $\mathrm{ht}(\pi_2') \leq \mathrm{ht}(\pi_2)$. The required derivation π' of $\Gamma \Rightarrow \Delta, \varphi(c)$ is:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2'\\ \Gamma \Rightarrow \Delta', \varphi(c), \psi & \Gamma \Rightarrow \Delta', \varphi(c), \chi\end{array}}{\Gamma \Rightarrow \Delta', \varphi(c), \psi \wedge \chi}\,\wedge\mathrm{R}$$

with $\mathrm{ht}(\pi') = \max(\mathrm{ht}(\pi_1'), \mathrm{ht}(\pi_2')) + 1 \leq \mathrm{ht}(\pi)$. □

Lemma seq.19 plays an important role in the proof of the cut-elimination theorem. But it can also be used to show the following:

Proposition seq.21. *The contraction rules* CL *and* CR *of* **G1c** *are height-preserving admissible in* **G3c**. *pt:seq:inv: prop:G3c-cont-adm*

Proof. We give the argument for CR and CL at the same time. Suppose π is a derivation of $\Gamma \Rightarrow \Delta, \varphi, \varphi$ or of $\varphi, \varphi, \Gamma \Rightarrow \Delta$ in **G3c**. We have to show that there is also a **G3c**-derivation π' of $\Gamma \Rightarrow \Delta, \varphi$ or $\varphi, \Gamma \Rightarrow \Delta$, respectively, with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$. We do this by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, π is just an axiom. But if $\Gamma \Rightarrow \Delta, \varphi, \varphi$ is an axiom of **G3c**, so is $\Gamma \Rightarrow \Delta, \varphi$: either some atomic χ is an element of both Γ and Δ, or φ is atomic and an element of Γ. The same reasoning applies if the end-sequent is $\varphi, \varphi, \Gamma \Rightarrow \Delta$.

If $n > 0$, π ends in some rule R. If φ is not principal in this last inference, we can obtain a derivation π' as in the proof of Lemma seq.19 by applying the inductive hypothesis to the immediate sub-derivations of π and applying the same rule R to the resulting derivation(s). The inductive hypothesis is that if there is a derivation of height $k < n$ of either $\Pi \Rightarrow \Lambda, \theta, \theta$ or $\theta, \theta, \Pi \Rightarrow \Lambda$, then there is a derivation of height $\leq k$ of $\Pi \Rightarrow \Lambda, \theta$ or $\theta, \Pi \Rightarrow \Lambda$, respectively. (Note that the inductive hypothesis applies if the contexts or contracted formula are different from Γ, Δ, or φ!)

For instance, suppose R is $\wedge$R and π ends in

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta', \psi, \varphi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \Gamma \Rightarrow \Delta', \chi, \varphi, \varphi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi, \varphi}\,\wedge\mathrm{R}$$

where $\Delta = \Delta', \psi \wedge \chi$. The inductive hypothesis yields derivations π'_1 and π'_2 of $\Gamma \Rightarrow \Delta', \psi, \varphi$ and $\Gamma \Rightarrow \Delta', \chi, \varphi$, respectively, with $\mathrm{ht}(\pi'_1) \leq \mathrm{ht}(\pi_1)$ and $\mathrm{ht}(\pi'_2) \leq \mathrm{ht}(\pi_2)$. The derivation π' is:

$$\dfrac{\begin{array}{c}\vdots\,\pi'_1\\ \Gamma \Rightarrow \Delta', \psi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi'_2\\ \Gamma \Rightarrow \Delta', \chi, \varphi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi}\,\wedge\mathrm{R}$$

If φ is principal in the last inference, we apply the inversion lemma (Lemma seq.19) to the immediate sub-derivations of π, then the inductive hypothesis, and then rule R again. For instance, suppose π proves $\Gamma \Rightarrow \Delta, \varphi, \varphi$, $\varphi \equiv (\psi \wedge \chi)$ and principal in the last inference. Then π looks as follows:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta, \psi \wedge \chi, \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \Gamma \Rightarrow \Delta, \psi \wedge \chi, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \wedge \chi, \psi \wedge \chi}\,\wedge\mathrm{R}$$

Apply Lemma seq.19 to π_1 to get a derivation π'_1 of $\Gamma \Rightarrow \Delta, \psi, \psi$. (It also yields a derivation of $\Gamma \Rightarrow \Delta, \chi, \psi$, but we don't need that.) Similarly, by applying Lemma seq.19 to π_2 we obtain a derivation π'_2 of $\Gamma \Rightarrow \Delta, \chi, \chi$. As the inversion of $\wedge$R is height-preserving, $\mathrm{ht}(\pi'_1) \leq \mathrm{ht}(\pi_1) < \mathrm{ht}(\pi)$ and $\mathrm{ht}(\pi'_2) \leq \mathrm{ht}(\pi_2) < \mathrm{ht}(\pi)$. Thus, the inductive hypothesis applies to π'_1 and π'_2: there are derivations π''_1 and π''_2 of $\Gamma \Rightarrow \Delta, \psi$ and $\Gamma \Rightarrow \Delta, \chi$, respectively, with $\mathrm{ht}(\pi''_1) \leq \mathrm{ht}(\pi'_1) \leq \mathrm{ht}(\pi_1)$ and $\mathrm{ht}(\pi''_2) \leq \mathrm{ht}(\pi'_2) \leq \mathrm{ht}(\pi_2)$. The derivation π' then is:

$$\dfrac{\begin{array}{c}\vdots\,\pi''_1\\ \Gamma \Rightarrow \Delta, \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi''_2\\ \Gamma \Rightarrow \Delta, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \wedge \chi}\,\wedge\mathrm{R}$$

with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$.

We must take special care for the quantifier rules in the cases where the formula is principal.

Suppose $\varphi \equiv \forall x\, \psi(x)$ is principal on the right. Then π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\Gamma \Rightarrow \Delta, \forall x\, \psi(x), \psi(a)}\,\forall\mathrm{R}$$

By Lemma seq.20, there is a derivation π_1' with height $\leq \mathrm{ht}(\pi_1)$ of $\Gamma \Rightarrow \Delta, \psi(c), \psi(a)$ for any c not occuring in $\Gamma \Rightarrow \Delta, \forall x\, \psi(x), \psi(a)$. We may assume that π_1' is regular; so by Lemma seq.9 the derivation $\pi_1'[a/c]$ is a derivation of $\Gamma \Rightarrow \Delta, \psi(a), \psi(a)$ also of height $\leq \mathrm{ht}(\pi_1)$. Now the inductive hypothesis applies and gives us a derivation π_1'' of $\Gamma \Rightarrow \Delta, \psi(a)$. Apply the $\forall$R rule to get π':

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1''\\ \vdots\end{array}}{\Gamma \Rightarrow \Delta, \psi(a)}\,\forall\mathrm{R}$$

with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$.

Now suppose $\varphi \equiv \exists x\, \psi(x)$ and principal on the right. Then π looks as follows:

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\Gamma \Rightarrow \Delta, \exists x\, \psi(x), \exists x\, \psi(x), \psi(t)}\,\exists\mathrm{R}$$

The inductive hypothesis applies to π_1 and there is a derivations π_1' of $\Gamma \Rightarrow \Delta, \exists x\, \psi(x), \psi(t)$. (Note that no inversion lemma is needed here!) The derivation π' then is:

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1'\\ \vdots\end{array}}{\Gamma \Rightarrow \Delta, \exists x\, \psi(x), \psi(t)}\,\exists\mathrm{R}$$

with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$. □

Problem seq.7. Sketch the proof of Proposition seq.21 for CL. Describe the cases where $\varphi \equiv (\psi \wedge \chi)$ and $\varphi \equiv (\psi \to \chi)$.

Problem seq.8. Sketch the proof of Proposition seq.21 for CL for the remaining quantifier cases, i.e., where $\varphi \equiv \forall x\, \psi(x)$ and $\varphi \equiv \exists x\, \psi(x)$.

seq.8 Translating between G1c and G3c

pt:seq:tr: sec

We mentioned that both **G1c** and **G3c** are sound and complete for classical logic, i.e., they each prove all and only the sequents true in every structure. In particular, they prove the same sequents. We are now in a position to prove this fact without the detour through the soundness and completeness theorems, but using purely proof-theoretically. Such a proof provides *transformations* of derivations in **G1c** into derivations in **G3c** and vice versa.

Proposition seq.22. *If* $\mathbf{G1c} \vdash S$ *then* $\mathbf{G3c} \vdash S$.

Proof. We show that if π is a derivation in **G1c**, then there is a derivation π' of S in **G3c**. We do this by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, then π is an axiom. The axiom $\bot \Rightarrow$ is also an axiom in **G3c** (take the context Γ, Δ to be empty). In **G3c**, axioms $\varphi \Rightarrow \varphi$ with arbitrary, not just atomic φ are allowed. But we proved in Proposition seq.7 that all such sequents are derivable in **G3c**.

If $n > 0$, π ends in a rule R. The inductive hypothesis guarantees that there are derivations of the premises of that rule in **G3c**, i.e., that the immediate sub-derivations have translations in **G3c**. If the rule R is also a rule in **G3c**, we apply it to the translated derivations of the premises. For those rules that are no also rules of **G3c**, we have already shown that they are at least admissible. So, if rule R was $\wedge$L or $\vee$R, apply Proposition seq.14. If it was a weakening rule, apply Proposition seq.17. If it was a contraction rule, apply Proposition seq.21. □

Proposition seq.23. *If* $\mathbf{G3c} \vdash S$ *then* $\mathbf{G1c} \vdash S$.

Proof. We again proceed by induction on the height of a derivation π of S. Every axiom of **G3c** can be derived in **G1c** from an axiom plus several applications of the WL and WR rules:

$$\dfrac{\dfrac{\chi \Rightarrow \chi}{\vdots}\,\mathrm{WL}}{\dfrac{\chi, \Gamma \Rightarrow \chi}{\vdots}\,\mathrm{WR}}\qquad \dfrac{\dfrac{\bot \Rightarrow}{\vdots}\,\mathrm{WL}}{\dfrac{\bot, \Gamma \Rightarrow}{\vdots}\,\mathrm{WR}}$$
$$\chi, \Gamma \Rightarrow \chi, \Delta \qquad \bot, \Gamma \Rightarrow \Delta$$

If π ends in a rule R, the inductive hypothesis yields derivations in **G1c** of the premises, and we can apply the same rule R to those derivations. The exceptions are the rules $\wedge$L and $\vee$R, which differ between **G1c** and **G3c**. The **G3c** versions are derivable in **G1c** as follows:

$$\dfrac{\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L}}{\dfrac{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\mathrm{CL}}\,\wedge\mathrm{L} \qquad \dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi, \psi}\,\vee\mathrm{R}}{\dfrac{\Gamma \Rightarrow \Delta, , \varphi \vee \psi, \varphi \vee \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\mathrm{CR}}\,\vee\mathrm{R}$$

□

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