seq.1 Rules and Derivations

pt:seq:rp: sec

We begin by giving some definitions and fixing some terminology.

Consider, for example, the two rules in **G3c** that allow deriving sequents containing $\varphi \wedge \psi$ from sequents containing φ and ψ:

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge\text{R}$$

The sequents on top of the line are called the *premises* and the sequent below the line the *conclusion*. The formulas in Γ and Δ in each rule are called the *context* or the *side formulas*. The formula in the conclusion that isn't part of the context is called the *principal* formula, and the formulas in the premises not in the context the *active* (or *auxiliary*) formulas.

The quantifier rules are slightly more complicated. E.g., the rules in **G1c** for $\forall$ are:

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta} \forall\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)} \forall\text{R}$$

In the $\forall$L rule, the active formula is $\varphi(t)$ where t is a closed term, i.e., a term without variables. Such an inference is correct—that is, it matches the schematic rule $\forall$L—if there is some formula φ, some variable x, and some closed term t such that the principal formula in the conclusion is $\forall x\, \varphi$ and the active formula in the premise is $\varphi[t/x]$. The $\forall$R rule works the same, except that instead of any term t the active formula in the premise must be $\varphi[a/x]$, where a is a constant symbol. This constant is called the *eigenvariable* of the $\forall$R inference. The inference is only correct if the lower sequent does not contain a, that is, if a does not occur in Γ, Δ, or φ. This is called the *eigenvariable condition*.

Sequent systems like **G1c** and **G3c** consists of a collection of such schematically specified rules together with some (also schematically specified) sequents that count as axioms. Derivations in such a system are finite trees of sequents that are inductively generated from axioms using the rules of inference. Each leaf is an axiom, and every other sequent follows from the sequents immediately above it by one of the inference rules of the system.

pt:seq:rp: defn:proof

Definition seq.1. *Derivations* in a sequent system are finite trees of sequents inductively as follows:

pt:seq:rp: defn:prf:axiom

1. Every axiom is a derivation.

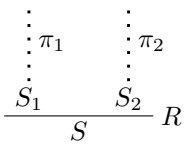
pt:seq:rp: defn:prf:one-prem

2. If π_1 is a derivation ending in a sequent S_1, and S is a sequent that follows from S_1 by a one-premise rule R, then

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ S_1\end{array}}{S}\, R$$

is a derivation.

3. If π_1 and π_2 are derivations ending in sequents S_1 and S_2, and S is a sequent that follows from S_1 andS_2 by a two-premise rule R, then pt:seq:rp: defn:prf:two-prem



is a derivation.

Definition seq.1 defines derivations as trees of sequents. We think of these trees as growing "upwards." The topmost (leaf) nodes of this tree are axioms and called *initial sequents*. The bottom-most sequent is called the *end-sequent*. If π is a derivation with end-sequent S we also write $\pi \vdash S$. If there is a derivation of a sequent S we write $\vdash S$, possibly indicating the system in which the proof exists on the left of the $\vdash$ symbol, e.g., $\mathbf{G1c} \vdash \varphi, \psi \Rightarrow \varphi \wedge \psi$. Every sequent in a derivation, with the exception of initial sequents, is the *conclusion* of an inference by some rule R. The one or two sequents standing immediately above a sequent in a derivation (i.e., the parents of the node) are called the *premises* of the inference.

The derivations used in the inductive construction of a derivation are called its *sub-derivations*. The sub-derivations ending in the premises of an inference are called the *immediate sub-derivations* of the inference.

It is often necessary to measure the complexity of derivations as a natural number. We could simply count the number of sequents in a derivation, but often a more useful measure is the height of a derivation: it is the maximal number of sequents between the end-sequent and an initial sequent. We can also give an inductive definition.

Definition seq.2. The *height* $\mathrm{ht}(\pi)$ of a derivation π is inductively defined as follows.

1. If π consists of a single axiom, $\mathrm{ht}(\pi) = 0$.
2. If π ends in an inference with one premise with immediate sub-derivation π_1, then $\mathrm{ht}(\pi) = \mathrm{ht}(\pi_1) + 1$.
3. If π ends in an inference with two premises with immediate sub-derivations π_1 and π_2, then $\mathrm{ht}(\pi) = \max(\mathrm{ht}(\pi_1), \mathrm{ht}(\pi_2)) + 1$.

If a sequent S has a derivation of height $\leq n$, we write $\vdash_n S$.

Example seq.3. In **G3c**, any sequent of the form $\chi, \Gamma \Rightarrow \Delta, \chi$ is an axiom, where χ must be atomic. Suppose θ and α are atomic sentences. Then $\theta, \alpha \Rightarrow \theta$ and $\theta, \alpha \Rightarrow \alpha$ are instances of the axiom $\chi, \Gamma \Rightarrow \Delta, \chi$. E.g., if we take $\chi \equiv \alpha$, $\Gamma = \{\theta\}$ and $\Delta = \emptyset$ in $\chi, \Gamma \Rightarrow \Delta, \chi$ we get exactly the sequent $\theta, \alpha \Rightarrow \alpha$. (Remember that we are dealing with multi*sets* and so $\theta, \alpha \Rightarrow \alpha$ and $\alpha, \theta \Rightarrow \theta$ are the same sequent.)

Since $\theta, \alpha \Rightarrow \theta$ is an instance of an axiom of **G3c**, it (by itself) counts as a derivation in **G3c**, and so does $\theta, \alpha \Rightarrow \alpha$ by Definition seq.1(1). Let's call them π_1 and π_2. By (3), we can take these two derivations and generate a new one (π_3) using the two-premise rule $\wedge$R:

$$\frac{\theta, \alpha \Rightarrow \theta \qquad \theta, \alpha \Rightarrow \alpha}{\theta, \alpha \Rightarrow \theta \wedge \alpha} \wedge\mathrm{R}$$

From π_3 in turn we obtain the derivation π_4

$$\pi_3 \left\{ \begin{array}{c} \theta, \alpha \Rightarrow \theta \qquad \theta, \alpha \Rightarrow \alpha \\ \hline \theta, \alpha \Rightarrow \theta \wedge \alpha \\ \hline \theta \wedge \alpha \Rightarrow \theta \wedge \alpha \end{array} \begin{array}{l} \wedge\mathrm{R} \\ \wedge\mathrm{L} \end{array} \right\} \pi_4$$

using the one-premise rule $\wedge$L (using Definition seq.1(2)). The immediate sub-derivation of the last inference in π_4 is π_3. The immediate sub-derivations of the $\wedge$R inference are π_1 and π_2. The sub-derivations of π_4 are all of π_1, π_2, π_3, and π_4 itself. We have $\mathrm{ht}(\pi_1) = \mathrm{ht}(\pi_2) = 0$, $\mathrm{ht}(\pi_3) = 1$ and $\mathrm{ht}(\pi_4) = 2$. We have shown that $\mathbf{G3c} \vdash_4 \theta \wedge \alpha \Rightarrow \theta \wedge \alpha$ for any atomic θ and α.

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Bibliography