seq.1 Regular Proofs and Substitution

pt:seq:qua:sec

The rules for the quantifiers introduce some complications owing to the "eigenvariable conditions" that apply to the ∃L and ∀R rules. Most of these complications can be dealt with by ensuring that the constant symbol a in these inferences is never reused. Let's introduce the following definition.

Definition seq.1. A derivation π in **G1c** or **G3c** is *regular* if

1. no eigenvariable a is used as the eigenvariable of more than one ∃L or ∀R inference, and
2. if a is the eigenvariable of a ∃L or ∀R inference, it only occurs above that inference in π.

pt:seq:qua:lem:subst-regular

Lemma seq.2. *If π is regular, ends in $\Gamma \Rightarrow \Delta$, c is a constant symbol not used as an eigenvariable in π, and t is a term not containing any eigenvariables of π, then $\pi[t/c]$ resulting from π by replacing every occurrence of c by t is a regular derivation. The end-sequent of $\pi[t/c]$ is $\Gamma[t/c] \Rightarrow \Delta[t/c]$, where $\Gamma[t/c] = \{\varphi(t) : \varphi(c) \in \Gamma\}$.*

Proof. By induction on $\mathrm{ht}(\pi)$. If $\mathrm{ht}(\pi) = 0$, it consists simply of an axiom. Then $\pi[t/c]$ is also an axiom, since any formula $\varphi(c)$ in it turns into $\varphi(t)$.

If $\mathrm{ht}(\pi) > 0$ we distinguish cases according to the last inference in π. The cases for structural rules (in **G1c**) and logical rules for propositional connectives are easy and left as exercises. We consider the cases where π ends in a quantifier inference.

1. If the last inference is ∀R, π is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \pi'\\ \vdots\end{array}}{\Gamma(c) \Rightarrow \Delta'(c), \varphi(a,c)}\,\forall\mathrm{R}$$
$$\Gamma(c) \Rightarrow \Delta'(c), \forall x\, \varphi(x,c)$$

   By induction hypothesis, $\pi'[t/c]$ is a regular derivation of $\Gamma(t) \Rightarrow \Delta'(t), \varphi(a,t)$. Since a is an eigenvariable of π, a does not occur in t. Therefore, the last inference in $\pi[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\\ \pi'[t/c]\\ \vdots\end{array}}{\Gamma(t) \Rightarrow \Delta'(t), \varphi(a,t)}\,\forall\mathrm{R}$$
$$\Gamma(t) \Rightarrow \Delta'(t), \forall x\, \varphi(x,t)$$

   is correct: the conclusion does not contain a.

2. If the last inference is ∀L in **G3c**, π is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \pi'\\ \vdots\end{array}\quad \varphi(s(c),c), \forall x\, \varphi(x,c), \Gamma(c) \Rightarrow \Delta'(c)}{\forall x\, \varphi(x,c), \Gamma(c) \Rightarrow \Delta'(c)}\,\forall\mathrm{R}$$

By induction hypothesis, $\pi'[t/c]$ is a regular derivation of $\varphi(s(t),t), \forall x\, \varphi(x,t), \Gamma(t) \Rightarrow \Delta'(t)$. The last inference in $\pi[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\\ \pi'[t/c]\\ \vdots\end{array}\quad \varphi(s(t),t), \forall x\, \varphi(x,t), \Gamma(t) \Rightarrow \Delta'(t)}{\forall x\, \varphi(x,t), \Gamma(t) \Rightarrow \Delta'(t)}\,\forall\mathrm{R}$$

is correct. The case for ∀L in **G1c** is similar, but slightly less complicated.

3. The last inference is ∃R or ∃L: exercise.

In each case, we added a sequent to the end of a regular derivation (or two regular derivations in the case of two-premise rules). The new sequent differs from the end-sequent of π only in that c was replaced by the term t. Moreover, the eigenvariables of π and Since t was assumed not to contain an eigenvariable of π and $\pi[t/c]$ are the same, the new end-sequent does not contain an eigenvariable of $\pi[t/c]$. Consequently, $\pi[t/c]$ is regular. □

Proposition seq.3. *If π is a derivation in* **G1c** *or* **G3c**, *then there is a regular derivation π' with the same structure (in particular, the same height) which differs from π only by replacing eigenvariables.*

Proof. For the purpose of the proof only, call an eigenvariable inference of π *clean* if its eigenvariable is not the eigenvariable of another inference nor does it occur anywhere in π except the immediate sub-derivation of the inference, and *dirty* otherwise. Let n be the number of dirty eigenvariable inferences in π. We show, by induction on n, that π can be transformed into the required regular derivation π'. If $n = 0$, all eigenvariable inferences in π are clean, so and π is regular, and there is nothing to prove. Otherwise, pick a highest eigenvariable inference, say ∀R. The sub-derivation π_1 of π ending in it is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \pi_2\\ \vdots\end{array}\quad \Gamma \Rightarrow \Delta, \varphi(c)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

Note that as c is an eigenvariable, it does not occur in Γ, Δ, or $\forall x\, \varphi(x)$. The proof π_2 is regular, since it contains no eigenvariable inferences at all. Let a

be a constant symbol not occuring in π. By Lemma seq.2, $\pi_2[a/c]$ is a regular proof of $\Gamma \Rightarrow \Delta, \varphi(a)$, to which we can apply $\forall$R. If we replace π_1 in π by the proof π_1',

$$\dfrac{\begin{array}{c}\vdots\\ \pi_2[a/c]\\ \vdots\\ \Gamma \Rightarrow \Delta, \varphi(a)\end{array}}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

we have replaced a dirty eigenvariable inference by a clean one, and the only difference is the replacement of the eigenvariable c by a. The resulting proof has $n-1$ dirty eigenvariable inferences, and the result follows from the inductive hypothesis. □

We will not appeal to the proposition just proved explicitly, but we will assume that all derivations considered are regular—if they are not, we can always replace eigenvariables to make them regular.

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Bibliography