seq.1 Examples of Proofs

pt:seq:ex:sec

To give derivations of sequents it is typically convenient to work your way upwards from the end-sequent by selecting a formula in it as the active formula and writing the premises of the corresponding rule on top of the sequent, then repeating the process until it leads to axioms. For instance, suppose we want to prove the sequent

$$(\chi \land \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)$$

Consider $\chi \to (\theta \to \alpha)$: it is of the form $\varphi \to \psi$ and occurs on the right of the sequent. Matching the entire sequent with the $\to$R rule of **G1c**,

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} \to\text{R}$$

suggests we should take $\Gamma = \{\chi \to (\theta \to \alpha)\}$, $\Delta = \emptyset$, $\varphi \equiv \chi$ and $\psi \equiv \theta \to \alpha$. So the last inference of a proof could be

$$\frac{\chi, (\chi \land \theta) \to \alpha \Rightarrow \theta \to \alpha}{(\chi \land \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)} \to\text{R}$$

If we repeat this idea, focusing on $\theta \to \alpha$ as active formula, we generate

$$\dfrac{\dfrac{\theta, \chi, (\chi \land \theta) \to \alpha \Rightarrow \alpha}{\chi, (\chi \land \theta) \to \alpha \Rightarrow \theta \to \alpha} \to\text{R}}{(\chi \land \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)} \to\text{R}$$

Now we have to focus on $(\chi \land \theta) \to \alpha$ and use the $\to$L rule of **G1c**,

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta} \to\text{L}$$

This leads us to

$$\dfrac{\dfrac{\dfrac{\theta, \chi \Rightarrow \alpha, \chi \land \theta \qquad \theta, \chi, \alpha \Rightarrow \alpha}{\theta, \chi, (\chi \land \theta) \to \alpha \Rightarrow \alpha} \to\text{L}}{\chi, (\chi \land \theta) \to \alpha \Rightarrow \theta \to \alpha} \to\text{R}}{(\chi \land \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)} \to\text{R}$$

Using the formula $\chi \land \theta$ in the left topmost sequent as principal and the $\land$R rule gives us

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\theta, \chi \Rightarrow \alpha, \chi \qquad \theta, \chi \Rightarrow \alpha, \theta}{\theta, \chi \Rightarrow \alpha, \chi \land \theta} \land\text{R} \qquad \theta, \chi, \alpha \Rightarrow \alpha}{\theta, \chi, (\chi \land \theta) \to \alpha \Rightarrow \alpha} \to\text{R}}{\chi, (\chi \land \theta) \to \alpha \Rightarrow \theta \to \alpha} \to\text{R}}{(\chi \land \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)} \to\text{R}$$

So far so good. Note that everything we did also works in **G3c**, since the rules we've made use of so far are the same in **G1c** and **G3c**. We've arrived at

a derivation fragment where all top sequents contain the same formula on the left as on the right. But they are not quite axioms yet.

In **G1c**, we have to give derivations of the topmost sequents from axioms of the form $\varphi \Rightarrow \varphi$. This is easily done using the weakening rules, e.g., the leftmost topmost sequent $\theta, \chi \Rightarrow \alpha, \chi$ can be derived as follows:

Overall, we have a derivation in **G1c** which looks like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\,\mathrm{WL}}{\theta, \chi \Rightarrow \alpha, \chi}\,\mathrm{WR} \quad \dfrac{\dfrac{\theta \Rightarrow \theta}{\theta, \chi \Rightarrow \theta}\,\mathrm{WL}}{\theta, \chi \Rightarrow \alpha, \theta}\,\mathrm{WR}}{\theta, \chi \Rightarrow \alpha, \chi \wedge \theta}\,\wedge\mathrm{R} \quad \dfrac{\dfrac{\alpha \Rightarrow \alpha}{\chi, \alpha \Rightarrow \chi}\,\mathrm{WL}}{\theta, \chi, \alpha \Rightarrow \alpha}\,\mathrm{WL}}{\theta, \chi, (\chi \wedge \theta) \to \alpha \Rightarrow \alpha}\,{\to}\mathrm{L}}{\chi, (\chi \wedge \theta) \to \alpha \Rightarrow \theta \to \alpha}\,{\to}\mathrm{R}}{(\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha)}\,{\to}\mathrm{R}$$

The structure of this proof (in particular, its height) is independent of what the formulas χ, θ, and α are. We can replace these symbols in the derivation above by any formulas and the result will still be a derivation in **G1c**. The proof in **G1c** is *schematic*.

In **G3c**, axioms are sequents of the form $\varphi, \Gamma \Rightarrow \Delta, \varphi$ where φ must be atomic. So although $\theta, \chi \Rightarrow \alpha, \chi$ looks like an axiom of **G3c** ($\Gamma = \{\theta\}$; $\Delta = \{\alpha\}$), it is only *really* an axiom if χ in fact is atomic. Our proof fragment is only a complete proof in **G3c** if χ, θ, and α are atomic formulas.

Although we haven't, strictly speaking, shown that

$$\mathbf{G1c} \vdash (\chi \wedge \theta) \to \alpha \Rightarrow \chi \to (\theta \to \alpha),$$

in practice this is all we need to do (and can do). That's because every sequent of the form $\varphi, \Gamma \Rightarrow \Delta, \varphi$ is provable in **G3c**, even if φ itself is not atomic. We can show this by induction on the depth $\mathrm{dp}(\varphi)$ of φ.

Definition seq.1. The *depth* $\mathrm{dp}(\varphi)$ of a formula is defined as follows: pt:seq:ex: defn:depth

1. If φ is atomic, $\mathrm{dp}(\varphi) = 0$.
2. If $\varphi \equiv \neg\psi$, $\varphi \equiv \forall x\, A$, or $\varphi \equiv \exists x\, A$, then $\mathrm{dp}(\varphi) = \mathrm{dp}(\psi) + 1$.
3. If $\varphi \equiv (\psi \wedge \chi)$, $(\psi \vee \chi)$, or $(\psi \to \chi)$, then $\mathrm{dp}(\varphi) = \max(\mathrm{dp}(\psi), \mathrm{dp}(\chi)) + 1$.

It is not hard to prove that $\mathrm{dp}(A(x)) = \mathrm{dp}(A(t))$ for any term t. The important fact for us is that in any rule, the depth of the principal formula is always greater than the depth of the active formulas. For instance, we have:

$$\begin{aligned} \mathrm{dp}(P(x)) = \mathrm{dp}(P(a)) &= 0, \\ \mathrm{dp}(\forall x\, P(x)) &= 1, \\ \mathrm{dp}((\forall x\, P(x) \to P(a))) &= \max(1, 0) + 1 = 2. \end{aligned}$$

We can use this to prove some results by induction on $\mathrm{dp}(\varphi)$, such as the following.

pt:seq:ex: prop:G1c-axioms **Proposition seq.2.** *For any φ, there is a derivation in* **G1c** *of $\varphi \Rightarrow \varphi$ in which all axioms are atomic.*

Proof. By induction on $\mathrm{dp}(\varphi)$. If $\mathrm{dp}(\varphi) = 0$ then φ is atomic and $\varphi \Rightarrow \varphi$ already is a derivation in **G1c** of the required form. If $\mathrm{dp}(\varphi) > 0$, φ is constructed from formulas of lower depth, e.g, $\varphi \equiv \neg\psi$, $\varphi \equiv (\psi \wedge \chi)$, or $\varphi \equiv \forall x\, A(x)$. The inductive hypothesis is that for all θ with $\mathrm{dp}(\theta) < \mathrm{dp}(\varphi)$, $\mathbf{G1c} \vdash \theta \Rightarrow \theta$ from atomic axioms.

If $\varphi \equiv \neg\psi$, then $\mathrm{dp}(\psi) < \mathrm{dp}(\varphi)$ and by inductive hypothesis, there is a derivation π_1 in **G1c** from atomic axioms of $\psi \Rightarrow \psi$. We can extend this to a derivation of $\neg\psi \Rightarrow \neg\psi$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\psi \Rightarrow \psi}\,\neg\mathrm{L}}{\neg\psi, \psi \Rightarrow}\,\neg\mathrm{R}}{\neg\psi \Rightarrow \neg\psi}$$

If $\varphi \equiv (\psi \wedge \chi)$, then $\mathrm{dp}(\psi) < \mathrm{dp}(\varphi)$ and $\mathrm{dp}(\chi) < \mathrm{dp}(\varphi)$. By inductive hypothesis, there are derivations π_1 and π_2 in **G1c** from atomic axioms of $\psi \Rightarrow \psi$ and $\chi \Rightarrow \chi$. We can extend these to a derivation of $\psi \wedge \chi \Rightarrow \psi \wedge \chi$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\psi, \Rightarrow \psi}\,\mathrm{WL}}{\psi, \chi, \Rightarrow \psi}\,\wedge\mathrm{L}}{\psi \wedge \chi, \chi \Rightarrow \psi}\,\wedge\mathrm{L} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_2\\ \vdots\end{array}}{\chi, \Rightarrow \chi}\,\mathrm{WL}}{\psi, \chi, \Rightarrow \chi}\,\wedge\mathrm{L}}{\psi \wedge \chi, \chi \Rightarrow \chi}\,\wedge\mathrm{L}}{\psi \wedge \chi, \psi \wedge \chi \Rightarrow \psi}\,\wedge\mathrm{R}}{\psi \wedge \chi \Rightarrow \psi \wedge \chi}\,\mathrm{CL}}{\psi \wedge \chi \Rightarrow \psi}$$

Now suppose $\varphi \equiv \forall x\, B(x)$. Take a constant symbol c that does not occur in $\forall x\, B(x)$. Then $\mathrm{dp}(B(c)) = \mathrm{dp}(\psi(x)) < \mathrm{dp}(\varphi)$ and by inductive hypothesis, there is a derivation π_1 in **G1c** from atomic axioms of $\psi(c) \Rightarrow \psi(c)$. We can extend this to a derivation of $\forall x\, B(x) \Rightarrow \forall x\, B(x)$ as follows:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\psi(c) \Rightarrow \psi(c)}\,\forall\mathrm{L}}{\forall x\, \psi(x) \Rightarrow \psi(c)}\,\forall\mathrm{R}}{\forall x\, \psi(x) \Rightarrow \forall x\, \psi(x)}$$

The remaining cases are left as exercises. □

Problem seq.1. Complete the proof of Proposition seq.2 by carrying out the cases where $\varphi \equiv (\psi \vee \chi)$, $\varphi \equiv (\psi \to \chi)$ and $\varphi \equiv \exists x\, B(x)$.

For our purposes, we could also have used the derivation

$$\dfrac{\dfrac{\vdots\ \pi_1}{\psi \Rightarrow \psi}}{\dfrac{\Rightarrow \psi, \neg\psi}{\neg\psi \Rightarrow \neg\psi}\,\neg\mathrm{L}}\,\neg\mathrm{R}$$

in the case where $\varphi \equiv \neg\psi$. However, there are sequent systems in which this wouldn't have worked. The intuitionistic sequent calculus **G1i** is like **G1c**, except it restricts all sequents to at most one formula on the right side of a sequent. If $\Delta = \emptyset$, then the first derivation would also be a derivation in **G1i**, but the second would not be, since one sequent contains both ψ and $\neg\psi$ on the right.

Note that even in **G1c** we could not have used an alternate order of rules in the case where $\varphi \equiv \forall x\, \psi(x)$. The following is *not* a derivation

$$\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{\psi(c) \Rightarrow \psi(c)}}{\psi(c) \Rightarrow \forall x\, \psi(x)}\,\forall\mathrm{R}}{\forall x\, \psi(x) \Rightarrow \forall x\, \psi(x)}\,\forall\mathrm{L}$$

since the eigenvariable condition on $\forall$R is violated.

Proposition seq.3. *Any sequent of the form $\varphi, \Gamma \Rightarrow \Delta, \varphi$ (with φ not necessarily atomic) is provable in* **G3c**. *pt:seq:ex: prop:G3c-axioms*

Proof. The proof is very similar to the one of Proposition seq.2, but we must be careful with the inductive hypothesis. The case where $n = \mathrm{dp}(\varphi) = 0$ is again easy. If $n = 0$, φ is atomic, and $\varphi, \Gamma \Rightarrow \Delta, \varphi$ is an axiom of **G3c**.

Now suppose $n > 0$. Again we distinguish cases. The inductive hypothesis is: for all D with $\mathrm{dp}(\theta) < n$, and all Π and Λ, **G3c** $\vdash \theta, \Pi \Rightarrow \Lambda, \theta$. Here is the case for $\varphi \equiv (\psi \wedge \chi)$. We use the inductive hypothesis twice: once for $\theta = \psi$, $\Pi = \chi, \Gamma$, and $\Lambda = \Delta$ to get a derivation π_1 of $\psi, \chi, \Gamma \Rightarrow \Delta, \psi$, and once for $\theta = \chi$, $\Pi = \psi, \Lambda$, and $\Lambda = \Delta$ to get a derivation π_2 of $\psi, \chi, \Gamma \Rightarrow \Delta, \chi$. Here is the derivation of $\psi \wedge \chi, \Gamma \Rightarrow \Delta, \psi \wedge \chi$:

$$\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{\psi, \chi, \Gamma \Rightarrow \Delta, \psi}}{\psi \wedge \chi, \Gamma \Rightarrow \Delta, \psi}\,\wedge\mathrm{L} \qquad \dfrac{\dfrac{\vdots\ \pi_2}{\psi, \chi, \Gamma \Rightarrow \Delta, \chi}}{\psi \wedge \chi, \Gamma \Rightarrow \Delta, \chi}\,\wedge\mathrm{L}}{\psi \wedge \chi, \Gamma \Rightarrow \Delta, \psi \wedge \chi}\,\wedge\mathrm{R}$$

For $\varphi \equiv \forall x\, \psi(x)$, pick a constant symbol not in $\forall x\, \psi(x)$, Γ, or Δ. Apply the inductive hypothesis for $\theta = \psi(c)$, $\Pi = \forall x\, \psi(x), \Gamma$ and $\Lambda = \Delta$. This gives a derivation π_1 of $\psi(c), \forall x\, \psi(x), \Gamma \Rightarrow \Delta, \psi(c)$, from which we obtain:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\end{array}}{\psi(c), \forall x\, \psi(x), \Gamma \Rightarrow \Delta, \psi(c)}\,\forall\mathrm{L}}{\forall x\, \psi(x), \Gamma \Rightarrow \Delta, \psi(c)}\,\forall\mathrm{R}$$

The eigenvariable condition on $\forall$R is satisfied by the way we chose c. □

Problem seq.2. Continue the proof of Proposition seq.3 by carrying out the cases where $\varphi \equiv (\psi \to \chi)$ and $\varphi \equiv \exists x\, B(x)$.

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Bibliography