seq.1 Invertibility of Rules

pt:seq:inv:sec

Definition seq.1. A rule R,

$$\frac{S_1 \qquad \dots \qquad S_n}{S}\, R$$

is *invertible* in a system if, whenever $\vdash S$ then $\vdash_{h_i} S_i$ for $i = 1, \dots, n$. It is *height-preserving invertible* (or *hp-invertible*) in a system if, whenever $\vdash_n S$ then $\vdash_n S_i$ for $i = 1, \dots, n$.

pt:seq:inv:lem:G3c-invert

Lemma seq.2. *The rules of* **G3c** *for* $\neg$, $\wedge$, $\vee$, *and* $\to$ *are height-preserving invertible, that is:*

1. $\neg$R*: If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \neg\varphi$, *then* $\mathbf{G3c} \vdash_n \varphi, \Gamma \Rightarrow \Delta$.
2. $\neg$L*: If* $\mathbf{G3c} \vdash_n \Gamma, \neg\varphi \Rightarrow \Delta$, *then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi$.
3. $\wedge$R*: If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi \wedge \psi$, *then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \psi$.
4. $\wedge$L*: If* $\mathbf{G3c} \vdash_n \Gamma, \varphi \wedge \psi \Rightarrow \Delta$, *then* $\mathbf{G3c} \vdash_n \varphi, \psi, \Gamma \Rightarrow \Delta$.
5. $\vee$R*: If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi \vee \psi$, *then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi, \psi$.
6. $\vee$L*: If* $\mathbf{G3c} \vdash_n \varphi \vee \psi, \Gamma \Rightarrow \Delta$, *then* $\mathbf{G3c} \vdash_n \varphi, \Gamma \Rightarrow \Delta$ *and* $\mathbf{G3c} \vdash_n \psi, \Gamma \Rightarrow \Delta$.
7. $\to$R*: If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi \to \psi$, *then* $\mathbf{G3c} \vdash_n \varphi, \Gamma \Rightarrow \Delta, \psi$.
8. $\to$L*: If* $\mathbf{G3c} \vdash_n \varphi \to \psi, \Gamma \Rightarrow \Delta$, *then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathbf{G3c} \vdash_n \psi, \Gamma \Rightarrow \Delta$.

Proof. We have to show that, for any rule R of **G3c**, if there is a derivation π of the conclusion S of R, then there are also derivations π_i of the premises S_i of R, with $\mathrm{ht}(\pi_i) \leq \mathrm{ht}(\pi)$. Consider the rule $\wedge$L: suppose there is a derivation π with end-sequent of the form $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$. We have to show that there is a derivation π' of $\varphi, \psi, \Gamma \Rightarrow \Delta$ with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$. We do this by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, π consists of just an axiom: $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$. In **G3c**, axioms are of the form $\chi, \Gamma \Rightarrow \Delta, \chi$ with χ atomic. In other words, χ can't be $\varphi \wedge \psi$. If $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$ is an axiom, then some atomic χ must be an element of both Γ and Δ. But then $\varphi, \psi, \Gamma \Rightarrow \Delta$ is also an axiom, and we have found our derivation π' (which is also of height 0).

If $n > 0$ it ends in an inference. We have to prove the claim for n, but may assume that it is true for any derivation of height $< n$, even if the context is different. In other words, for any $k < n$ and any Π and Λ, if $\varphi \wedge \psi, \Pi \Rightarrow \Lambda$ has a derivation of height k, then there is a derivation of height $\leq k$ of $\varphi, \psi, \Pi \Rightarrow \Lambda$.

We have two cases: either $\varphi \wedge \psi$ on the left is principal, or it is not. If it is principal, the last inference is $\wedge$L and the immediate sub-derivation π_1 ends in $\varphi, \psi, \Gamma \Rightarrow \Delta$. The result holds in this case as $\mathrm{ht}(\pi_1) < \mathrm{ht}(\pi)$.

If $\varphi \wedge \psi$ is not principal, some other formula is and $\varphi \wedge \psi$ belongs to the context of the last inference. Then the inductive hypothesis applies to the immediate sub-derivations of the last rule R, which yields a derivation π_1' or two derivations π_1' and π_2' of height no greater than the height(s) of the immediate sub-derivations of π. The end-sequents of π_1' and π_2' contain φ, ψ in the context on the left instead of $\varphi \wedge \psi$. Apply the rule R to obtain the proof π'. For instance, suppose π ends in $\to$L:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \varphi \wedge \psi, \Gamma' \Rightarrow \Delta, \chi & \varphi \wedge \psi, \Gamma', \theta \Rightarrow \Delta\end{array}}{\varphi \wedge \psi, \Gamma', \chi \to \theta \Rightarrow \Delta} \to\mathrm{L}$$

Here $\chi \to \theta$ on the left is principal, and the context on the left is $\varphi \wedge \psi, \Gamma'$ (i.e., $\Gamma = \Gamma', \chi \to \theta$). The inductive hypothesis yields derivations π_1' and π_2' of $\varphi \wedge \psi, \Gamma' \Rightarrow \Delta, \chi$ and $\varphi \wedge \psi, \Gamma', \theta \Rightarrow \Delta$, respectively. These two sequents can again serve as premises for the $\to$L rule to get π':

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2'\\ \varphi, \psi, \Gamma' \Rightarrow \Delta, \chi & \varphi, \psi, \Gamma', \theta \Rightarrow \Delta\end{array}}{\varphi, \psi, \Gamma', \chi \to \theta \Rightarrow \Delta} \to\mathrm{L}$$

We have

$$\begin{aligned}\mathrm{ht}(\pi) &= \max(\mathrm{ht}(\pi_1), \mathrm{ht}(\pi_2)) + 1\\ \mathrm{ht}(\pi') &= \max(\mathrm{ht}(\pi_1'), \mathrm{ht}(\pi_2')) + 1\end{aligned}$$

by definition, and

$$\begin{aligned}\mathrm{ht}(\pi_1') &\leq \mathrm{ht}(\pi_1)\\ \mathrm{ht}(\pi_2') &\leq \mathrm{ht}(\pi_2)\end{aligned}$$

by inductive hypothesis. So,

$$\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi).$$ □

Problem seq.1. Give the proof of Lemma seq.2 for $\to$R.

Lemma seq.3. *The rules* $\forall$R *and* $\exists$L *are height-preserving invertible in the following sense:* *pt:seq:inv: lem:invert-quant*

1. *If* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \forall x\, \varphi(x)$ *then* $\mathbf{G3c} \vdash_n \Gamma \Rightarrow \Delta, \varphi(c)$, *and*

2. *if* $\mathbf{G3c} \vdash_n \exists x\, \varphi(x), \Gamma \Rightarrow \Delta$ *then* $\mathbf{G3c} \vdash_n \varphi(c), \Gamma \Rightarrow \Delta$,

for any c not occuring in Γ, Δ, and $\forall x\, \varphi(x)$ or $\exists x\, \varphi(x)$, respectively.

Proof. We show (1) and leave (2) as an exercise. The argument is similar to the proof of Lemma seq.2. In the inductive step, there are again two cases. The case where $\forall x\, \varphi(x)$ is principal again is quick: the premise of the last $\forall$R inference is of the required form $\Gamma \Rightarrow \Delta, \varphi(a)$ and the immediate sub-derivation π_1 ending in it has $\mathrm{ht}(\pi_1) < \mathrm{ht}(\pi)$. Moreover, the eigenvariable condition gives us that a does not occur in Γ, Δ, or $\forall x\, \varphi(x)$. Since we may assume that π_1 is regular, $\pi_1[c/a]$ is a derivation of the same height of $\Gamma \Rightarrow \Delta, \varphi(c)$ by **??**.

In the second case, $\forall x\, \varphi(x)$ is not principal in the last inference; some other formula is. We again consider a case by way of example. Suppose the last inference is $\wedge$R. Some formula in Δ is of the form $\psi \wedge \chi$ and is principal. The derivation π ends in

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta', \forall x\, \varphi(x), \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \Gamma \Rightarrow \Delta', \forall x\, \varphi(x), \chi\end{array}}{\Gamma \Rightarrow \Delta', \forall x\, \varphi(x), \psi \wedge \chi}\,\wedge\mathrm{R}$$

where $\Delta = \Delta', \psi \wedge \chi$. Let c be a constant symbol that does not appear in Δ; then clearly it also does not occur in Δ', ψ or in Δ', χ. So the inductive hypothesis applies to π_1 and π_2; we have derivations π_1' and π_2' with $\mathrm{ht}(\pi_1') \leq \mathrm{ht}(\pi_1)$ and $\mathrm{ht}(\pi_2') \leq \mathrm{ht}(\pi_2)$. The required derivation π' of $\Gamma \Rightarrow \Delta, \varphi(c)$ is:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1'\\ \Gamma \Rightarrow \Delta', \varphi(c), \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2'\\ \Gamma \Rightarrow \Delta', \varphi(c), \chi\end{array}}{\Gamma \Rightarrow \Delta', \varphi(c), \psi \wedge \chi}\,\wedge\mathrm{R}$$

with $\mathrm{ht}(\pi') = \max(\mathrm{ht}(\pi_1'), \mathrm{ht}(\pi_2')) + 1 \leq \mathrm{ht}(\pi)$. □

Lemma seq.2 plays an important role in the proof of the cut-elimination theorem. But it can also be used to show the following:

pt:seq:inv: prop:G3c-cont-adm

Proposition seq.4. *The contraction rules* CL *and* CR *of* **G1c** *are height-preserving admissible in* **G3c**.

Proof. We give the argument for CR and CL at the same time. Suppose π is a derivation of $\Gamma \Rightarrow \Delta, \varphi, \varphi$ or of $\varphi, \varphi, \Gamma \Rightarrow \Delta$ in **G3c**. We have to show that there is also a **G3c**-derivation π' of $\Gamma \Rightarrow \Delta, \varphi$ or $\varphi, \Gamma \Rightarrow \Delta$, respectively, with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$. We do this by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, π is just an axiom. But if $\Gamma \Rightarrow \Delta, \varphi, \varphi$ is an axiom of **G3c**, so is $\Gamma \Rightarrow \Delta, \varphi$: either some atomic χ is an element of both Γ and Δ, or φ is

atomic and an element of Γ. The same reasoning applies if the end-sequent is $\varphi, \varphi, \Gamma \Rightarrow \Delta$.

If $n > 0$, π ends in some rule R. If φ is not principal in this last inference, we can obtain a derivation π' as in the proof of Lemma seq.2 by applying the inductive hypothesis to the immediate sub-derivations of π and applying the same rule R to the resulting derivation(s). The inductive hypothesis is that if there is a derivation of height $k < n$ of either $\Pi \Rightarrow \Lambda, \theta, \theta$ or $\theta, \theta, \Pi \Rightarrow \Lambda$, then there is a derivation of height $\leq k$ of $\Pi \Rightarrow \Lambda, \theta$ or $\theta, \Pi \Rightarrow \Lambda$, respectively. (Note that the inductive hypothesis applies if the contexts or contracted formula are different from Γ, Δ, or φ!)

For instance, suppose R is $\wedge$R and π ends in

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta', \psi, \varphi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \Gamma \Rightarrow \Delta', \chi, \varphi, \varphi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi, \varphi}\,\wedge\mathrm{R}$$

where $\Delta = \Delta', \psi \wedge \chi$. The inductive hypothesis yields derivations π'_1 and π'_2 of $\Gamma \Rightarrow \Delta', \psi, \varphi$ and $\Gamma \Rightarrow \Delta', \chi, \varphi$, respectively, with $\mathrm{ht}(\pi'_1) \leq \mathrm{ht}(\pi_1)$ and $\mathrm{ht}(\pi'_2) \leq \mathrm{ht}(\pi_2)$. The derivation π' is:

$$\dfrac{\begin{array}{c}\vdots\,\pi'_1\\ \Gamma \Rightarrow \Delta', \psi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi'_2\\ \Gamma \Rightarrow \Delta', \chi, \varphi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi}\,\wedge\mathrm{R}$$

If φ is principal in the last inference, we apply the inversion lemma (Lemma seq.2) to the immediate sub-derivations of π, then the inductive hypothesis, and then rule R again. For instance, suppose π proves $\Gamma \Rightarrow \Delta, \varphi, \varphi$, $\varphi \equiv (\psi \wedge \chi)$ and principal in the last inference. Then π looks as follows:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta, \psi \wedge \chi, \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \Gamma \Rightarrow \Delta, \psi \wedge \chi, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \wedge \chi, \psi \wedge \chi}\,\wedge\mathrm{R}$$

Apply Lemma seq.2 to π_1 to get a derivation π'_1 of $\Gamma \Rightarrow \Delta, \psi, \psi$. (It also yields a derivation of $\Gamma \Rightarrow \Delta, \chi, \psi$, but we don't need that.) Similarly, by applying Lemma seq.2 to π_2 we obtain a derivation π'_2 of $\Gamma \Rightarrow \Delta, \chi, \chi$. As the inversion of $\wedge$R is height-preserving, $\mathrm{ht}(\pi'_1) \leq \mathrm{ht}(\pi_1) < \mathrm{ht}(\pi)$ and $\mathrm{ht}(\pi'_2) \leq \mathrm{ht}(\pi_2) < \mathrm{ht}(\pi)$. Thus, the inductive hypothesis applies to π'_1 and π'_2: there are derivations π''_1 and π''_2 of $\Gamma \Rightarrow \Delta, \psi$ and $\Gamma \Rightarrow \Delta, \chi$, respectively, with $\mathrm{ht}(\pi''_1) \leq \mathrm{ht}(\pi'_1) \leq \mathrm{ht}(\pi_1)$

and $\mathrm{ht}(\pi_2'') \leq \mathrm{ht}(\pi_2') \leq \mathrm{ht}(\pi_2)$. The derivation π' then is:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1'' & \vdots\,\pi_2'' \\ \Gamma \Rightarrow \Delta, \psi & \Gamma \Rightarrow \Delta, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \wedge \chi}\,\wedge\mathrm{R}$$

with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$.

We must take special care for the quantifier rules in the cases where the formula is principal.

Suppose $\varphi \equiv \forall x\, \psi(x)$ is principal on the right. Then π ends in

$$\dfrac{\begin{array}{c}\vdots\,\pi_1 \\ \Gamma \Rightarrow \Delta, \forall x\, \psi(x), \psi(a)\end{array}}{\Gamma \Rightarrow \Delta, \forall x\, \psi(x), \forall x\, \psi(x)}\,\forall\mathrm{R}$$

By Lemma seq.3, there is a derivation π_1' with height $\leq \mathrm{ht}(\pi_1)$ of $\Gamma \Rightarrow \Delta, \psi(c), \psi(a)$ for any c not occuring in $\Gamma \Rightarrow \Delta, \forall x\, \psi(x), \psi(a)$. We may assume that π_1' is regular; so by **??** the derivation $\pi_1'[a/c]$ is a derivation of $\Gamma \Rightarrow \Delta, \psi(a), \psi(a)$ also of height $\leq \mathrm{ht}(\pi_1)$. Now the inductive hypothesis applies and gives us a derivation π_1'' of $\Gamma \Rightarrow \Delta, \psi(a)$. Apply the $\forall$R rule to get π':

$$\dfrac{\begin{array}{c}\vdots\,\pi_1'' \\ \Gamma \Rightarrow \Delta, \psi(a)\end{array}}{\Gamma \Rightarrow \Delta, \forall x\, \psi(x)}\,\forall\mathrm{R}$$

with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$.

Now suppose $\varphi \equiv \exists x\, \psi(x)$ and principal on the right. Then π looks as follows:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1 \\ \Gamma \Rightarrow \Delta, \exists x\, \psi(x), \exists x\, \psi(x), \psi(t)\end{array}}{\Gamma \Rightarrow \Delta, \exists x\, \psi(x), \exists x\, \psi(x)}\,\exists\mathrm{R}$$

The inductive hypothesis applies to π_1 and there is a derivations π_1' of $\Gamma \Rightarrow \Delta, \exists x\, \psi(x), \psi(t)$. (Note that no inversion lemma is needed here!) The derivation π' then is:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1' \\ \Gamma \Rightarrow \Delta, \exists x\, \psi(x), \psi(t)\end{array}}{\Gamma \Rightarrow \Delta, \exists x\, \psi(x)}\,\exists\mathrm{R}$$

with $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$. □

Problem seq.2. Sketch the proof of Proposition seq.4 for CL. Describe the cases where $\varphi \equiv (\psi \wedge \chi)$ and $\varphi \equiv (\psi \to \chi)$.

Problem seq.3. Sketch the proof of Proposition seq.4 for CL for the remaining quantifier cases, i.e., where $\varphi \equiv \forall x\, \psi(x)$ and $\varphi \equiv \exists x\, \psi(x)$.

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Bibliography