seq.1 Introduction

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Sequent calculi are derivation systems that were introduced in the 1930s by Gerhard Gentzen. They were intended less as a practical method for deriving things. Rather, Gentzen used them in order to be able to prove certain results about the possible structure of derivations and their property. These kinds of questions are exactly the questions with which proof theory is concerned.

Sequent calculi operate on *sequents*, as the name suggests, and not on formulas. A sequent is a pair of sequences or multisets of formulas. Gentzen's original systems (**LK** and **LJ**) used sequences; proof theorists now prefer to use multisets. A multiset is like a set, except that a multiset can contain an element more than once. Just as in sets, the order of elements does not matter, however. So the expressions $\{\varphi, \varphi, \psi\}$ and $\{\varphi, \psi, \varphi\}$ describe the same multiset, but that multiset is different from $\{\varphi, \psi\}$ and from $\{\varphi, \varphi, \psi, \psi\}$, since the latter contain different numbers of φ and ψ than the former.

Traditionally, multisets of formulas in proof theory are written using uppercase Greek letters. So when we write Γ, Δ, Π, Λ, or Θ, we intend these to be multisets of formulas of whatever logic we are working with. Traditionally also, instead of writing $\langle \Gamma, \Delta \rangle$ proof theorists separate the two components by a sequent symbol; we use $\Rightarrow$.

Definition seq.1 (Sequent). A *sequent* is an expression of the form

$$\Gamma \Rightarrow \Delta$$

where Γ and Δ are finite (possibly empty) multisets of formulas. Γ is called the *antecedent*, while Δ is the *succedent*.

Let γ be the conjunction of the formulas in Γ (taking the empty conjunction to be $\top$) and θ be the disjunction of the formulas in Δ (taking the empty disjunction to be $\bot$). Then the sequent $\Gamma \Rightarrow \Delta$ is true (in some structure $\mathfrak{M}$) iff $\gamma \to \theta$ is true. In classical logic, this amounts to the same as saying that either one of the formulas in Γ is false, or one of the formulas in Δ is true.

If Γ is a multiset of formulas, we write Γ, φ (or φ, Γ) for the result of adding φ to Γ. If Δ is also a multiset of formulas, then Γ, Δ is the multiset union of the two sequences—if Γ contains φ with *multiplicity* n (i.e., it contains n copies of φ) and Δ with multiplicity m, then Γ, Δ contains φ with multiplicity $n + m$. If the multiset on one side of a sequent is empty, we usually just leave a blank. For instance, $\Rightarrow \varphi$ is the sequent with an empty antecendent and the succedent contains φ with multiplicity 1.

A derivation in the sequent calculus is a tree of sequents, where the topmost (or initial) sequents are axioms in the system considered, and if a sequent stands below one or two other sequents, it must follow correctly by a rule of inference in the system. We will first consider two different sequent systems for classical logic, **G1c** and **G3c**. Their axioms and rules are given in Tables 1 and 2.

The systems are subtly different, and these differences mean that they have different proof theoretic properties. **G1c** has axioms $\varphi \Rightarrow \varphi$ with no other

Axioms

$$\varphi \Rightarrow \varphi \qquad\qquad \bot \Rightarrow$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{WL} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{WR}$$

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{CL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{CR}$$

Logical Rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\,\neg\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\,\neg\mathrm{R}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\,\wedge\mathrm{R}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\,\vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\,\to\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\,\to\mathrm{R}$$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\exists\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\,\exists\mathrm{R}$$

Table 1: Rules of **G1c**. In ∀R and ∃L, the constant symbol a must not occur in the conclusion sequent. Γ and Δ are multisets of formulas.

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tab:G1c

Axioms

$$\chi, \Gamma \Rightarrow \Delta, \chi \qquad\qquad \bot, \Gamma \Rightarrow \Delta$$

Logical Rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\,\neg\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\,\neg\mathrm{R}$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\,\wedge\mathrm{R}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\,\vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\,\to\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\,\to\mathrm{R}$$

$$\frac{\varphi(t), \forall x\, \varphi(x) \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\exists\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x), \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\,\exists\mathrm{R}$$

Table 2: Rules of **G3c**. In ∀R and ∃L, the constant symbol a must not occur in the conclusion sequent. Γ and Δ are multisets of formulas. In axioms, the formula χ must be atomic.

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tab:G3c

formulas allowed. **G3c** has axioms of the form $\chi, \Gamma \Rightarrow \Delta, \chi$ with arbitrary "side" formulas in Γ and Δ, but the formula χ appearing on both sides must be *atomic*. **G1c** has two versions each of the ∧L and ∨R rules whereas **G3c** has just one each. And **G1c** has additional "structural rules" called "contraction" (CL, CR) and "weakening" (WL, WR).

The systems **G1c** and **G3c** happen to be sound and complete for classical logic. In other words, every sequent derivable in these systems is true in every structure, and every sequent that is true in every structure has a derivation. So the question *whether* a sequent has a derivation can also be answered by other methods of logic (e.g., those of model theory). And since both systems derive exactly the sequents true in every structure, they in particular prove the same sequents.

Proof theory, however, is not just concerned with *whether* something is provable, but *how*. Proof theorists consider questions such as "Can a certain rule always be avoided?," "Can this rule be added to the system without resulting in new sequents becoming provable?," or "Can derivations in one system be turned into derivations in another?" If the answer is "yes" to such a question, the proof of this fact is often proved by induction on the complexity of derivations, or, equivalently, by induction on the structure of derivations. This

yields not just a proof of the fact (e.g., if **G1c** proves a sequent then **G3c** proves the same sequent), but a procedure (e.g., of turning derivations in **G1c** into derivations in **G3c**).

A central result in proof theory is the *cut-elimination theorem*. It shows that we can add the cut rule

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \text{Cut}$$

to the sequent systems without changing which sequents are derivable. Moreover, the proof provides a method that transforms any derivation using the rules of **G1c** (or **G3c**) together with Cut into one that only uses the rules of **G1c** (or **G3c**). In other words, this procedure "eliminates" the Cut rule from derivations using it; hence the name.

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Bibliography