seq.1 Interpretation of Rules

pt:seq:irl:sec

Recall the interpretation of a sequent: A sequent $\Gamma \Rightarrow \Delta$ is true in a structure $\mathfrak{M}$ if either at least one $\varphi \in \Gamma$ is false or at least one $\varphi \in \Delta$ is true. We can read the logical rules of **G3c** as codifying the truth conditions for their principal formulas. Consider $\to$R. Its principal formula is $\varphi \to \psi$. It is true if either φ is false or ψ is true. Hence the sequent $\Rightarrow \varphi \to \psi$ is true iff $\varphi \Rightarrow \psi$ is true. If we add the context formulas Γ, Δ to the antecedents and succedents of these sequents, we get exactly the conclusion and premise of the $\to$R rule. The situation is similar for $\to$L. $\varphi \to \psi$ is false if φ is true and ψ is false. So $\varphi \to \psi \Rightarrow$ is true iff both $\Rightarrow \varphi$ and $\psi \Rightarrow$ are true. The conclusion of $\to$L is equivalent to the conjunction of its premises. This pattern ensures that the rules of **G3c** are sound (if all premises are true, the conclusion is true). They also ensure completeness.

In fact, the truth conditions for any truth-functional connective can be described in this way as a set of sequents. This follows from the fact that every truth-function in classical logic can be described by a formula in conjunctive normal form: a conjunction of disjunctions of atomic and negated atomic formulas. For instance, take $\varphi \oplus \psi$ to be true iff exactly one of φ and ψ is true, i.e., "exclusive or." Then $\varphi \oplus \psi$ is true iff $(\varphi \vee \psi) \wedge (\neg\varphi \vee \neg\psi)$ is, and it is false if $(\neg\varphi \vee \psi) \wedge (\varphi \vee \neg\psi)$ is true. Hence, we could introduce sequent calculus rules for $\oplus$ as

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta, \varphi}{\varphi \oplus \psi, \Gamma \Rightarrow \Delta} \oplus\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi \qquad \varphi, \psi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi \oplus \psi} \oplus\text{R}$$

and these rules would also be sound and complete. Their conclusions are true iff all premises are true.

Problem seq.1. Suppose $\varphi \downarrow \psi$ is true iff neither φ nor ψ is true ("NOR"). What would its **G3c** rules be? (Find two sequents which are true at the same time iff $\varphi \downarrow \psi$ is true. That gives you the $\downarrow$R rule. Then find one sequent that is true iff $\varphi \downarrow \psi$ is false. That gives you the $\downarrow$L rule.)

In **G3c**, each connective and quantifier has exactly two rules: a left rule and a right rule. In **G1c**, however, there are two $\wedge$L and two $\vee$R rules. The reason is that Gentzen's original system **LK**, on which **G1c** is modeled, had a variant **LJ** for an important non-classical logic, viz., intuitionistic logic. To get a system that was sound and complete for intuitionistic logic, the succedents of any sequent in **LJ** is restricted to contain at most one formula. This makes it impossible to use the $\vee$R rule of **G3c**, as its premise contains both φ and ψ in the succedent. Gentzen thus used a pair of $\vee$R rules for **LJ**, and used the same for **LK**. The intuitionistic version **G1i** similarly is just **G1c** with all succedents restricted to at most one formula. (**G3c** also has an intuitionistic version, but it some of its rules differ from the corresponding rules in **G3c**. For instance, it, too, uses a pair of $\vee$R rules.)

That the structural rules of **G1c** are sound is easy to see. For instance, if $\Gamma \Rightarrow \Delta$ contains a false formula on the right or a true formula on the left, so does $\Gamma \Rightarrow \Delta, \varphi$: the same one, whether φ is true or false is irrelevant. So if the premise of WR is true, so is its conclusion. Note that here the converse is not true. The conclusion may be true because φ is true, and then the premise is not guaranteed to be true.

Why have structural rules at all? Contraction (CR, CL) is needed for instance to derive $\Rightarrow \varphi \vee \neg\varphi$. If you start with $\varphi \Rightarrow \varphi$ you first get $\Rightarrow \varphi, \neg\varphi$ by $\neg$R. Then you can use the two versions of $\vee$R in succession, once to get $\varphi \vee \neg\varphi$ from $\neg\varphi$, and once to get $\varphi \vee \neg\varphi$ from φ. What you get is $\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi$. So to get $\Rightarrow \varphi \vee \neg\varphi$ in **G1c** you need to be able to "contract" two occurrences of the same formula in a derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\,\neg\mathrm{R}}{\Rightarrow \varphi, \varphi \vee \neg\varphi}\,\vee\mathrm{R}}{\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi}\,\vee\mathrm{R}}{\Rightarrow \varphi \vee \neg\varphi}\,\mathrm{CR}$$

There are in fact valid sequents which you could not derive in **G1c** without weakening or contraction. For instance, a derivation of the sequent $\varphi \Rightarrow \psi \to \varphi$ in **G1c** has to end in $\to$R (since only $\psi \to \varphi$ can be principal), and that requires $\psi, \varphi \Rightarrow \varphi$ as a premise. This, in turn, requires WL to get it from the axiom $\varphi \Rightarrow \varphi$:

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\psi, \varphi \Rightarrow \varphi}\,\mathrm{WL}}{\varphi \Rightarrow \psi \to \varphi}\,\to\mathrm{R}$$

In **G3c** the contraction and weakening rules are not needed at all. Weakening is built into the axioms. Contraction can mostly be avoided by merging the two copies of the context Γ, Δ in rules with two premises into one. Both **G1c** and **G3c** have such "context sharing" rules. In one-premise rules, contraction is only needed for $\vee$R, $\wedge$L, $\exists$R, and $\forall$L. The versions of these rules in **G3c** make contraction entirely superfluous. There is a system **G2c** (see Table 1) in which the two-premise rules have independent contexts. In **G2c** contraction is even more important than in **G1c**.

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Bibliography

Axioms

$$\varphi \Rightarrow \varphi \qquad\qquad \bot \Rightarrow$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{WL} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{WR}$$

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{CL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{CR}$$

Logical Rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\,\neg\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\,\neg\mathrm{R}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \Gamma_2 \Rightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \wedge \psi}\,\wedge\mathrm{R}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\mathrm{L}$$

$$\frac{\varphi, \Gamma_1 \Rightarrow \Delta_1 \qquad \psi, \Gamma_2 \Rightarrow \Delta_2}{\varphi \vee \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,\vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\,\vee\mathrm{R}$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \psi, \Gamma_2 \Rightarrow \Delta_2}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,\to\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\,\to\mathrm{R}$$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\,\forall\mathrm{R}$$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\,\exists\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\,\exists\mathrm{R}$$

Table 1: Rules of **G2c**. In ∀R and ∃L, the constant symbol a must not occur in the conclusion sequent. Γ and Δ are multisets of formulas.

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tab:G2c