seq.1 Admissible and Derivable Rules

pt:seq:adm: sec

Many results of proof theory concern—or use—results about whether everything that can be derived using a specific rule can also be derived without that rule. The cut-elimination theorem is the most well-known example. It shows that any sequent that can be derived using the rules of a sequent system with Cut can also be derived without Cut. Because this kind of property is so important, it has a name.

Definition seq.1. A rule R is *admissible* in a sequent system if any sequent that can be proved in that system together with the rule R can also be proved without R.

pt:seq:adm: ex:land-deriv

Example seq.2. The rule $\wedge$L takes different forms in **G1c** and **G3c**. In **G1c** there are two versions of the rule, one with the left conjunct φ of the principal formula $\varphi \wedge \psi$ in the premise, and one with the right conjunct ψ:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_1 \qquad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_2$$

By contrast, **G3c** has only one rule:

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_3$$

We might now ask, "Is **G3c**'s rule $\wedge\mathrm{L}_3$ admissible in **G1c**?" It is: we can directly simulate any inference using $\wedge\mathrm{L}_3$ using only the rules of **G1c**. In addition to $\wedge\mathrm{L}_1$ and $\wedge\mathrm{L}_2$ we'll also need the structural rule CL:

$$\dfrac{\dfrac{\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_1}{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\mathrm{L}_2}{\varphi \wedge \psi \Gamma \Rightarrow \Delta} \mathrm{CL}$$

Simulating a inferences using one rule using other rules is one way to show that a rule is admissible. But it is not the only way, and some rules are admissible but can't be so simulated. Rules that can be are called *derivable*.

Definition seq.3. A rule R,

$$\frac{S_1 \qquad \dots \qquad S_n}{S} R$$

is called *derivable* in a system if there is a schematic derivation of the sequent S using the rules and axioms of the system together with additional initial sequents of the forms of $S_1, \dots, S_n$.

The derivation given in Example seq.2 shows that $\wedge\mathrm{L}_3$ is a derivable rule in **G1c**, as that derivation is a schematic derivation of the conclusion of $\wedge\mathrm{L}_3$ from the premises of $\wedge\mathrm{L}_3$ as initial sequents using only rules of **G1c**. (It didn't use any of the axioms of **G1c**.)

Proposition seq.4. *The rules* ∧L *and* ∨R *of* **G3c** *are derivable in* **G1c**. pt:seq:adm: prop:lor-G3-adm

Proof. The proof for ∧L is given in Example seq.2. The proof for ∨R is left as an exercise. □

Problem seq.1. Show that the rule ∨R of **G3c** is derivable in **G1c**.

Problem seq.2. Suppose ↔ is a defined logical operator: $\varphi \leftrightarrow \psi$ is short for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. Show that the rules

1. $$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \leftrightarrow \psi} \leftrightarrow\text{L}$$
2. $$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \leftrightarrow \psi} \leftrightarrow\text{R}$$

are derivable in (a) **G1c** and (b) **G3c**.

Example seq.5. The rule WR of **G1c**, pt:seq:adm: ex:weak-adm

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,\text{WR}$$

is admissible in **G3c**. The idea of the proof is simple: suppose there is a derivation π of the premise $\Gamma \Rightarrow \Delta$ in **G3c**. Add the formula φ to the succedent of every sequent in π. Axioms remain axioms; correct inferences are still correct inferences. The end-formula of the new derivation is $\Gamma \Rightarrow \Delta, \varphi$.

The rule WR is, however, not derivable in **G3c**. For suppose it were, i.e., suppose one could find a derivation of $\Gamma \Rightarrow \Delta, \varphi$ using only the axioms and rules of **G3c** plus possibly the additional initial sequent $\Gamma \Rightarrow \Delta$. If this derivation is, as would be required for WR to be derivable, purely schematic, we could turn it into a derivation of $\Rightarrow \varphi$ by deleting Γ and Δ. This would turn the additional initial sequents $\Gamma \Rightarrow \Delta$ into the empty sequent $\Rightarrow$. Since the empty sequent contains no formulas at all, but all rules of **G3c** require at least one active formula in their premises, the empty sequent can't be a premise of any inference in this schematic derivation. Consequently, the derivation can only be schematic if it doesn't actually use the additional sequent $\Gamma \Rightarrow \Delta$ as an initial sequent. In other words, it would be a schematic derivation of $\Gamma \Rightarrow \Delta, \varphi$ using only the axioms and rules of **G3c** and we would have shown that any sequent of this form has a derivation in **G3c**. But this is impossible, since **G3c** is sound and so can only derive sequents that are true in every structure. Taking $\Gamma = \Delta = \emptyset$ and $\varphi \equiv \bot$, **G3c** would have to derive, e.g., the sequent $\Rightarrow \bot$, which it does not.

Let's give a nice inductive proof of the result that weakening is an admissible rule in **G3c**. In fact, a slightly stronger result holds, as the intuitive argument shows: by adding φ to the right side of every sequent of π, we do not change the structure of the derivation, in particular, we do not increase the height. Since this is important, it deserves its own definition.

Definition seq.6. A rule R,

$$\frac{S_1 \qquad \dots \qquad S_n}{S}\, R$$

is *height-preserving admissible* (or *hp-admissible*) in a system if, whenever $\vdash_{h_i} S_i$ for $i = 1, \dots, n$, then $\vdash_m S$ with $m = \max(h_1, \dots, h_n)$.

Note that it is enough for a rule to be height-preserving admissable that, whenever the premises S_i have derivations π_i with height $h_i = \mathrm{ht}(\pi_i)$, there is a derivation π of the conclusion of height $\mathrm{ht}(\pi)$ no greater than the maximum of the h_i. In particular, if there is only one premise S_1, it is enough that $\mathrm{ht}(\pi) \leq \mathrm{ht}(\pi_1)$. In the case of WR and WL we actually have $\mathrm{ht}(\pi) = \mathrm{ht}(\pi_1)$, but that is not necessary.

pt:seq:adm: prop:weak-G3c-adm

Proposition seq.7. *The rules* WR *and* WL *of* **G1c** *are height-preserving admissible in* **G3c**.

Proof. We have to show that if $\mathbf{G3c} \vdash \Gamma \Rightarrow \Delta$ with a derivation π of height $\mathrm{ht}(\pi) = n$ then there is a **G3c**-derivation π' of $\Gamma \Rightarrow \Delta, \varphi$ with $\mathrm{ht}(\pi') = n$. We prove this by induction on n. If $n = 0$, then π is just the axiom $\Gamma \Rightarrow \Delta$ (i.e., some atomic χ occurs in both Γ and Δ), and so is $\Gamma \Rightarrow \Delta, \varphi$. If $n = \mathrm{ht}(\pi) > 0$, the derivation π has a last inference. We distinguish cases according to which rule was used. We give just one example: suppose the last inference was $\wedge$R. Then $\Delta = \Delta', \psi \wedge \chi$ and the immediate sub-derivations π_1 and π_2 of the last inference have end-sequents $\Gamma \Rightarrow \Delta', \psi$ and $\Gamma \Rightarrow \Delta', \chi$, respectively. In other words, π looks like this:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma \Rightarrow \Delta', \psi & \Gamma \Rightarrow \Delta', \chi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi}\,\wedge\mathrm{R}$$

We also have that $n = \max(\mathrm{ht}(\pi_1), \mathrm{ht}(\pi_2)) + 1$. So $\mathrm{ht}(\pi_1) < n$ and $\mathrm{ht}(\pi_2) < n$, and the inductive hypothesis applies: there are derivations π_1' and π_2' of $\Gamma \Rightarrow \Delta', \psi, \varphi$ and $\Gamma \Rightarrow \Delta', \chi, \varphi$. Using the $\wedge$R rule, we obtain a derivation π' of $\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi$:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2'\\ \Gamma \Rightarrow \Delta', \psi, \varphi & \Gamma \Rightarrow \Delta', \chi, \varphi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi}\,\wedge\mathrm{R}$$

We have $\mathrm{ht}(\pi') = \max(\mathrm{ht}(\pi_1'), \mathrm{ht}(\pi_2')) + 1 = \max(\mathrm{ht}(\pi_1), \mathrm{ht}(\pi_2)) + 1 = \mathrm{ht}(\pi)$.□

Proposition seq.7 is extremely useful. We'll introduce a notation for repeated application of it: if π is a derivation of $\Gamma \Rightarrow \Delta$ then $\pi[\Pi \Rightarrow \Lambda]$ is the

result of applying admissibility of weakening on the left with all the formulas in Π and admissibility of weakening on the right with all formulas in Λ. It is a derivation of $\Gamma, \Pi \Rightarrow \Delta, \Lambda$.

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Bibliography