pty.1 Type Preservation

int:pty:tpr:sec

The reduction and permutation conversions considered can also be defined directly for the systems **tN2i** and **tN3i**. The system **tN3i** defines the type of a λ-term N relative to the context Γ as the formula φ such that $\Gamma \Rightarrow N : \varphi$ has a derivation in **tN3i**. The fact that the β-reduction rules for λ-terms exactly mirror the reduction and permutation conversions of derivations has an important consequence: the type of a λ-term relative to a context Γ remains the same after β-conversion. This property is called *type preservation*.

Proposition pty.1. *If N has type φ relative to Γ and $N \to N'$, then N' has type φ relative to Γ.*

Proof. By induction on N and the height of derivations δ of $\Gamma \Rightarrow N : \varphi$ we can easily establish the following: If M is a sub-term of N, then δ contains a sub-derivation δ_1 ending in $\Gamma' \Rightarrow M : \psi$. If M is a redex, then a conversion applies to δ_1. By applying the reduction to the δ_1 we obtain a derivation δ_1' of $\Gamma' \Rightarrow M' : \psi$. By inspection of the conversions of **tN3i**, M' is the reductum of M. Replacing δ_1 in δ by δ_1' results in a derivation δ' of $\Gamma \Rightarrow N' : \varphi$, where N' results from N by replacing the subterm M by M'.

For instance, consider the reduction conversion for $\wedge$Intro followed by $\wedge$Elim:

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma \Rightarrow N_1 : \psi_1 & \Gamma \Rightarrow N_2 : \psi_2\end{array}}{\Gamma \Rightarrow \langle N_1, N_2\rangle : \psi_1 \wedge \psi_2}\,\wedge\text{Intro}}{\Gamma \Rightarrow \pi_i(\langle N_1, N_2\rangle) : \psi_i}\,\wedge\text{Elim}_i \qquad \to \qquad \begin{array}{c}\vdots \\ \Gamma \Rightarrow N_i : \psi_i\end{array}$$

We see that the proof term of the derivation on the right, i.e., δ_1' is N_i. It is exactly the result of applying β-conversion to the proof term $\pi_i(\langle N_1, N_2\rangle)$ of the derivation on the left (δ_1).

Or consider a $\to$Intro followed by an $\to$Elim inference, and the reduction

conversion applied to it:

$$\dfrac{\dfrac{\begin{array}{c}\delta_2\\ \vdots\\ x:\psi_1,\Gamma \Rightarrow N:\psi_2\end{array}}{\Gamma \Rightarrow \lambda x^{\psi_1}.N:\psi_1\to\psi_2}\to\text{Intro} \qquad \begin{array}{c}\delta_3\\ \vdots\\ \Gamma \Rightarrow M:\psi_1\end{array}}{\Gamma \Rightarrow (\lambda x^{\psi_1}.NM):\psi_2}\to\text{Elim} \quad \to \quad \begin{array}{c}\delta_3\\ \vdots\\ \Gamma \Rightarrow M:\psi_1\\ \vdots\\ \delta_2[\delta_3/x:\psi_1]\\ \vdots\\ \Gamma \Rightarrow N[M/x]:\psi_2\end{array}$$

Again the proof term of the derivation on the right is exactly the result of β-reducing the proof term of the derivation on the left. Of course one would have to carefuly verify (by induction on the height of δ_2) that if we replace all axioms of the form $\Gamma' \Rightarrow x:\psi_1$ in δ_2 by the derivation δ_3 of $\Gamma \Rightarrow M:\psi_1$ actually results in a derivation $\delta_2[\delta_3/x:\psi_1]$ of $\Gamma \Rightarrow N[M/x]:\psi_2$. □

The tight correspondence between derivations and λ-terms has another consequence. If N is a λ-term of type φ relative to a context Γ and contains a redex M (i.e., it is not a normal form), then the corresponding **tN3i** proof δ of $\Gamma \Rightarrow N:\varphi$ contains a sub-derivation (ending in $\Gamma' \Rightarrow M:\psi$) to which a conversion can be applied. If $N \to N'$ by replacing M by its reductum, the corresponding result of applying the reduction to δ is a derivation of $\Gamma \Rightarrow N':\varphi$. Hence every λ-term not in normal form β-reduces to another. This property is called *progress*.

Progress together with type preservation ensure that β-reduction of λ-terms never "gets stuck": if a term is well-typed and isn't in normal form, it reduces to another well-typed term (and that term is of the same type).

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Bibliography