pty.1 Recovering Derivations from Proof Terms

int:pty:tp:sec

Now let us consider the other direction: translating a proof term N back to a derivation in **N2i**. In general, it is only possible to do this relative to a context that contains pairs $x : \varphi$ for every variable x that occurs free in the given proof term. But let's start with an example where there are no free variables, namely the term

$$\lambda x^{\varphi\wedge\psi}.\,\langle\pi_2(x),\pi_1(x)\rangle$$

It is of the form $\lambda x^{\varphi\wedge\psi}.\,N_1$. Since the only rule of **tN2i** that produces a term of this form is $\to$Intro, we know that the derivation encoded by N has to end with $\to$Intro, i.e., it ends in

$$\frac{x : \varphi\wedge\psi \;\Rightarrow\; \langle\pi_2(x),\pi_1(x)\rangle : \chi}{\Rightarrow\; \lambda x^{\varphi\wedge\psi}.\,\langle\pi_2(x),\pi_1(x)\rangle : (\varphi\wedge\psi)\to\chi}\to\text{Intro}$$

We don't yet know what χ is, but we do know that the antecedent is $\varphi\wedge\psi$, and that the context of the premise has to include $x : \varphi\wedge\psi$.

Now the proof term associated with the premise is already determined: $\langle\pi_2(x),\pi_1(x)\rangle$. The derivation *it* encodes must end with $\wedge$Intro. We still don't know what χ is, but because it is the conclusion of $\wedge$Intro, we know it must be a conjunction, i.e., $\chi \equiv (\chi_1\wedge\chi_2)$. We continue the reconstruction:

$$\dfrac{\dfrac{x : \varphi\wedge\psi \;\Rightarrow\; \pi_2(x) : \chi_1 \qquad x : \varphi\wedge\psi \;\Rightarrow\; \pi_1(x) : \chi_2}{x : \varphi\wedge\psi \;\Rightarrow\; \langle\pi_2(x),\pi_1(x)\rangle : \chi_1\wedge\chi_2}\wedge\text{Intro}}{\Rightarrow\; \lambda x^{\varphi\wedge\psi}.\,\langle\pi_2(x),\pi_1(x)\rangle : (\varphi\wedge\psi)\to(\chi_1\wedge\chi_2)}\to\text{Intro}$$

The proof term $\pi_2(x)$ means that top sequent on the left was obtained by $\wedge\text{Elim}_2$, i.e., the premise is of the form $\theta\wedge\chi_1$. On the right, we can determine that the inference leading to χ_2 must be $\wedge\text{Elim}_1$, and that the premise is of the form $\chi_2\wedge\alpha$.

$$\dfrac{\dfrac{\dfrac{x : \varphi\wedge\psi \;\Rightarrow\; x : \theta\wedge\chi_1}{x : \varphi\wedge\psi \;\Rightarrow\; \pi_2(x) : \chi_1}\wedge\text{Elim}_2 \qquad \dfrac{x : \varphi\wedge\psi \;\Rightarrow\; x : \chi_2\wedge\alpha}{x : \varphi\wedge\psi \;\Rightarrow\; \pi_1(x) : \chi_2}\wedge\text{Elim}_1}{x : \varphi\wedge\psi \;\Rightarrow\; \langle\pi_2(x),\pi_1(x)\rangle : \chi_1\wedge\chi_2}\wedge\text{Intro}}{\Rightarrow\; \lambda x^{\varphi\wedge\psi}.\,\langle\pi_2(x),\pi_1(x)\rangle : (\varphi\wedge\psi)\to(\chi_1\wedge\chi_2)}\to\text{Intro}$$

Since the proof terms of the topmost sequents are now variables, we know that they must be axioms. This determines χ_1, χ_2, θ, and α: $\theta \equiv \varphi$ and $\chi_1 \equiv \psi$ or else the left sequent is not an axiom, and $\chi_2 \equiv \varphi$ and $\alpha \equiv \psi$, or else the right sequent isn't. We have recovered the proof fully:

$$\dfrac{\dfrac{\dfrac{x : \varphi\wedge\psi \;\Rightarrow\; x : \varphi\wedge\psi}{x : \varphi\wedge\psi \;\Rightarrow\; \pi_2(x) : \psi}\wedge\text{Elim}_2 \qquad \dfrac{x : \varphi\wedge\psi \;\Rightarrow\; x : \varphi\wedge\psi}{x : \varphi\wedge\psi \;\Rightarrow\; \pi_1(x) : \varphi}\wedge\text{Elim}_1}{x : \varphi\wedge\psi \;\Rightarrow\; \langle\pi_2(x),\pi_1(x)\rangle : \psi\wedge\varphi}\wedge\text{Intro}}{\Rightarrow\; \lambda x^{\varphi\wedge\psi}.\,\langle\pi_2(x),\pi_1(x)\rangle : (\varphi\wedge\psi)\to(\psi\wedge\varphi)}\to\text{Intro}$$

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Bibliography