pty.1 Converting Derivations to Proof Terms

int:pty:pt: sec

The process of converting natural deduction derivations to proof terms, i.e., translating derivations in **N2i** to derivations in **tN2i**, is extremely straightforward. It simply amounts to writing a proof term to the left of each formula on the right of a sequent in the derivation, resulting in sequents of the form $\Gamma \Rightarrow M : \varphi$. We'll then say that M *witnesses* φ in context Γ. Which proof term to write is entirely determined by the rules of **tN2i**.

Proposition pty.1. *Every derivation δ of $\Gamma \Rightarrow \varphi$ in* **N2i** *corresponds to a derivation δ' of $\Gamma \Rightarrow N : \varphi$ in* **tN2i**. *The proof term N is uniquely determined by δ.*

Proof. We proceed by induction on the height of δ. If δ is the axiom $x : \varphi \Rightarrow \varphi$, then δ' is the axiom $x : \varphi \Rightarrow x : \varphi$. If δ ends in an inference rule, the inductive hypothesis provides proof terms for each premise, and **tN2i**-derivations of sequents including these proof terms corresponding to each premise. We apply the corresponding rule of **tN2i**; the corresponding proof term is determined by the **tN2i** inference rule. □

Example pty.2. Consider the very simple derivation in **N1i** of $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$:

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^x}{\psi}\wedge\text{Elim}_2 \quad \dfrac{[\varphi \wedge \psi]^x}{\varphi}\wedge\text{Elim}_1}{\psi \wedge \varphi}\wedge\text{Intro}}{(\varphi \wedge \psi) \to (\psi \wedge \varphi)}\,{\scriptstyle x}\,\to\text{Intro}$$

In **N2i** it looks like this:

$$\dfrac{\dfrac{\dfrac{x : \varphi \wedge \psi \;\Rightarrow\; \varphi \wedge \psi}{x : \varphi \wedge \psi \;\Rightarrow\; \psi}\wedge\text{Elim}_2 \quad \dfrac{x : \varphi \wedge \psi \;\Rightarrow\; \varphi \wedge \psi}{x : \varphi \wedge \psi \;\Rightarrow\; \varphi}\wedge\text{Elim}_1}{x : \varphi \wedge \psi \;\Rightarrow\; \psi \wedge \varphi}\wedge\text{Intro}}{\Rightarrow\; (\varphi \wedge \psi) \to (\psi \wedge \varphi)}\to\text{Intro}$$

We assign proof terms from the top down. The proof terms for the axioms are just the same as the variable in the context, x. The proof terms associated with $\wedge\text{Elim}_i$ are then determined as $\pi_2(x)$ and $\pi_1(x)$. When we apply $\wedge$Intro, we combine these two proof terms: $\langle \pi_2(x), \pi_1(x) \rangle$. Finally, the $\to$Intro rule applies a λ-abstraction, binding x and recording that x labelled the assumption $\varphi \wedge \psi$: $\lambda x^{\varphi \wedge \psi}.\, \langle \pi_2(x), \pi_1(x) \rangle$. The derivation in **tN2i** then is:

$$\dfrac{\dfrac{\dfrac{x : \varphi \wedge \psi \;\Rightarrow\; x : \varphi \wedge \psi}{x : \varphi \wedge \psi \;\Rightarrow\; \pi_2(x) : \psi}\wedge\text{Elim}_2 \quad \dfrac{x : \varphi \wedge \psi \;\Rightarrow\; x : \varphi \wedge \psi}{x : \varphi \wedge \psi \;\Rightarrow\; \pi_1(x) : \varphi}\wedge\text{Elim}_1}{x : \varphi \wedge \psi \;\Rightarrow\; \langle \pi_2(x), \pi_1(x) \rangle : \psi \wedge \varphi}\wedge\text{Intro}}{\Rightarrow\; \lambda x^{\varphi \wedge \psi}.\, \langle \pi_2(x), \pi_1(x) \rangle : (\varphi \wedge \psi) \to (\psi \wedge \varphi)}\to\text{Intro}$$

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Bibliography