pty.1 Proof Terms

int:pty:ter: sec

We label assumptions in **N1i** with variables (e.g., φ^x) as discharge labels. In **N2i**, every sequent $\Gamma \Rightarrow \varphi$ has a list of variable–formula pairs in the context Γ on the left. Thus, in **N2i**, every sequent in a derivation carries not only the information what the derivation so far has derived (i.e., φ), but also which assumptions are open at each point (i.e., Γ). What if we also included in each sequent the information *how* φ was derived. To this end, we can define a *term-labelled* system **tN2i**, in which sequents have the form $\Gamma \Rightarrow N : \varphi$. Γ and φ are as before: a set of pairs of the form $x : \psi$ and the formula the sub-derivation ending in the sequent has shown to follow from the assumptions in Γ. N will be a certain term that encodes the derivation ending in the sequent.

The terms we assign to each sequent are built from variables x, y, ..., using function symbols which encode the inferences carried out so far. We need a separate function symbol for each inference rule. The function symbol corresponding to introduction rules are called "constructors" and those corresponding to elimination rules "destructors." Each function symbol takes as many arguments as the rule has premises. For instance, the function symbols for $\wedge$Intro and $\wedge\text{Elim}_1$ might be $c^\wedge$ (2-place) and $d_1^\wedge$ (1-place). In **tN2i** these rules then become

$$\frac{\Gamma_1 \Rightarrow N : \varphi \qquad \Gamma_2 \Rightarrow M : \psi}{\Gamma_1, \Gamma_2 \Rightarrow c^\wedge(N, m) : \varphi \wedge \psi} \wedge\text{Intro} \qquad \frac{\Gamma \Rightarrow N : \varphi \wedge \psi}{\Gamma \Rightarrow d_1^\wedge(N) : \varphi} \wedge\text{Elim}_1$$

If an inference rule discharges an assumption, i.e., the rule has a premise that indicates a labelled formula $x : \varphi$ in a premise that is not in the context of the conclusion, we have to indicate this in the proof term as well. The function symbols associated with such rules *bind* the corresponding variables. For instance, the $\to$Intro rule goes from the premise $x : \varphi, \Gamma \Rightarrow \psi$ to $\Gamma \Rightarrow \varphi \to \psi$. The constructor $c^\to$ takes one argument, but also must indicate that the variable x becomes bound. This is the proof term's way of expressing that the assumption with label x has been discharged. For instance, if the proof term for the premise is N, we could write the proof term for the conclusion as $c^\to([x]N)$ and the rule as

$$\frac{x : \varphi, \Gamma \Rightarrow N : \psi}{\Gamma \Rightarrow c^\to([x]N) : \varphi \to \psi} \to\text{Intro}$$

In order to be able to fully reconstruct a derivation, we need some additional information. If the premise formulas of a rule don't fully determine what the conclusion formula is, we have to add that information to the term. This is the case in $\to$Intro, so we will write $c^\to([x : \varphi]N)$. It is also the case in $\vee$Intro and $\bot_I$, so we use function symbols that carry that information, e.g.,

$$\frac{\Gamma \Rightarrow N : \varphi}{\Gamma \Rightarrow c_1^{\vee\psi} : \varphi \vee \psi} \vee\text{Intro}_1 \qquad \frac{\Gamma \Rightarrow N : \bot}{\Gamma \Rightarrow d_\varphi^\bot(N) : \varphi} \bot_I$$

With these ideas it is possible to construct a sequent-style natural deduction system in which every sequent is of the form $\Gamma \Rightarrow N : \varphi$, where N is a *proof term*.

It turns out that there is a close connection between such proof terms and terms in the so-called *typed lambda calculus*. To respect this connection, we'll use symbols used in the literature on the lambda calculus rather than the generic constructor and destructor symbols above. E.g., instead of writing $c^{\to}([x:\varphi]N)$ we write $\lambda x^{\varphi}.\,N$, instead of $d^{\to}(N,M)$ we write (NM), $\langle N,M\rangle$ instead of $c^{\wedge}(N,M)$, and $\pi_i(N)$ instead of $d_i^{\wedge}(N)$.

Definition pty.1. Proof terms are inductively generated by the following rules:

1. Every variable x is a proof term (Axioms).
2. If x is a variable, φ is a formula, and N is a proof term, then $\lambda x^{\varphi}.\,N$ is also a proof term ($\to$Intro).
3. If N and M are proof terms, then (NM) is also a proof term ($\to$Elim).
4. If N and M are proof terms, then $\langle N,M\rangle$ is a proof term ($\wedge$Intro).
5. If N is a proof term, then $\pi_i(N)$ is also a proof term, where i is 1 or 2 ($\wedge\text{Elim}_i$).
6. If N is a proof term, and φ is a formula, then $\iota_i^{\varphi}(M)$ is a proof term, where i is 1 or 2 ($\vee\text{Intro}_i$).
7. If M, N_1, N_2 are proof terms, and x_1, x_2 are variables, then $\delta\, M\, x_1.N_1\, x_2.N_2$ is a proof term ($\vee$Elim).
8. If N is a proof term and φ is a formula, then $\varepsilon^{\varphi}(N)$ is proof term ($\bot_I$).

Not every proof term is the proof term of a derivation. For instance, a term $\langle N,M\rangle$ encodes a proof ending in a $\wedge$Intro inference, so its end-formula is a conjunction $\varphi\wedge\psi$. This can't be the major premise of a $\to$Elim rule, so $(\langle N,M\rangle P)$ can't be the proof term of a correct proof. Only those proof terms that follow the rules of **tN2ip** are *correct* proof terms. These rules are given in Table 1.

Photo Credits

Bibliography

Axioms:

$$x : \varphi \Rightarrow x : \varphi$$

Inference Rules:

$$\frac{x : \varphi, \Gamma \;\Rightarrow\; N : \psi}{\Gamma \;\Rightarrow\; \lambda x^{\varphi}.\, N : \varphi \to \psi}\;{\to}\text{Intro}$$

$$\frac{\Gamma_1 \;\Rightarrow\; N : \varphi \to \psi \qquad \Gamma_2 \;\Rightarrow\; M : \varphi}{\Gamma_1, \Gamma_2 \;\Rightarrow\; (NM) : \psi}\;{\to}\text{Elim}$$

$$\frac{\Gamma_1 \;\Rightarrow\; N_1 : \varphi \qquad \Gamma_2 \;\Rightarrow\; N_2 : \psi}{\Gamma_1, \Gamma_2 \;\Rightarrow\; \langle N_1, N_2 \rangle : \varphi \wedge \psi}\;{\wedge}\text{Intro}$$

$$\frac{\Gamma \;\Rightarrow\; N : \varphi \wedge \psi}{\Gamma \;\Rightarrow\; \pi_1(N) : \varphi}\;{\wedge}\text{Elim}_1 \qquad\qquad \frac{\Gamma \;\Rightarrow\; N : \varphi \wedge \psi}{\Gamma \;\Rightarrow\; \pi_2(N) : \psi}\;{\wedge}\text{Elim}_2$$

$$\frac{\Gamma \;\Rightarrow\; N : \varphi}{\Gamma \;\Rightarrow\; \iota_1^{\psi}(N) : \varphi \vee \psi}\;{\vee}\text{Intro}_1 \qquad \frac{\Gamma \;\Rightarrow\; N : \psi}{\Gamma \;\Rightarrow\; \iota_2^{\varphi}(N) : \varphi \vee \psi}\;{\vee}\text{Intro}_2$$

$$\frac{\Gamma_1 \;\Rightarrow\; M : \varphi \vee \psi \qquad x : \varphi, \Gamma_2 \;\Rightarrow\; N_1 : \chi \qquad y : \psi, \Gamma_3 \;\Rightarrow\; N_2 : \chi}{\Gamma_1, \Gamma_2, \Gamma_3 \;\Rightarrow\; \delta\, M\, x.N_1\, y.N_2 : \chi}\;{\vee}\text{Elim}$$

$$\frac{\Gamma \;\Rightarrow\; N : \bot}{\Gamma \;\Rightarrow\; \varepsilon^{\varphi}(N) : \varphi}\;\bot_I$$

Table 1: Rules of the propositional term-labelled intuitionistic natural deduction system **tN2ip**.

int:pty:ter:
tab:tN2ip