nor.1 Translating Between Normal N2i-Derivations and G2i

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Derivations in **N2i** can be translated to derivations in **G2i** and vice versa. First of all, the result of translating cut-free **2i** derivations results in normal **N2i**-derivations.

To state the result we need a bit of terminology to let us relate antecendents of sequents in **N2i** and in **G2i** more easily: a set Γ' of labelled formulas *corresponds to* the multiset Γ of unlabelled formulas if, for every formula φ, the number of elements of Γ' of the form $x : \varphi$ is less than or equal to the multiplicity of φ in Γ.

Corollary nor.1. *If* $\mathbf{G2i} \vdash \Gamma \Rightarrow \Delta$ *then there is a normal* **N2c**-*derivation* δ *of* $\Gamma' \Rightarrow \chi$*, where the set* Γ' *of labelled formulas corresponds to the multiset* Γ*, i.e., for every formula* φ*, the number of elements of* Γ' *of the form* $x : \varphi$ *is less than or equal to the multiplicity of* φ *(i.e., the number of copies of* φ*) in* Γ*.*

Proof. By inspection of the proof of **??**: we add introduction rules to the end of derivations, and elimination rules only at the top of derivations, so never introduce an introduction rule above an elimination rule. □

The translation of the Cut rule given in **??**, however, can introduce cuts in the **N2i**-derivation δ. If a **G2i** derivation ends in Cut with immediate subderivations π_1 and π_2 of $\Gamma_1 \Rightarrow \varphi$ and $\varphi, \Gamma_2 \Rightarrow \Delta_2$, respectively, we graft the translation δ_1 of π_1 onto the axioms $x : \varphi \Rightarrow \varphi$ occurring in the translation δ_2 of π_2. If δ_1 ends in an introduction rule with principal formula φ and the axiom $x : \varphi \Rightarrow \varphi$ in δ_2 is the major premise of an elimination rule, this would result in a cut in δ.

It is also possible to translate derivations in **N2i** to derivations in **G2i**. In the proof of **??**, the translation of ∧Elim, →Elim, ¬Elim, ∀Elim results in Cut inferences in the resulting **G2i** derivation. If the **N2i**-derivation we start from, this can, however, be avoided.

Let's call a list $S_1, \dots, S_n$ of occurrences of sequents in a **N2i**-derivation δ a *branch* if S_1 is an axiom, and whenever S_i is a major premise of an inference, S_{i+1} is its conclusion. In other words, a branch traces sequents downwards in δ from axioms, but stops at minor premises of elimination rules. A *main branch* of δ is a branch where S_n is the end-sequent. Every derivation δ must have at least one main branch, since every rule has at least one major premise (only ∧Intro has two).

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Proposition nor.2. *If* δ *is a normal* **N2c** *or* **N2i**-*derivation of* $\Gamma \Rightarrow \varphi$ *then there is a (cut-free)* **G2c**-*derivation (***G2i**-*derivation)* π *of* $\Gamma' \Rightarrow \Delta$*, where* Γ' *is the multiset of formulas resulting from* Γ *by removing labels, and* $\Delta = \{\varphi\}$ *if* φ *is not* $\bot$*, and* $\Delta = \emptyset$ *if it is.*

Proof. This time, induction is on the number of inferences in the derivation δ of $\Gamma \Rightarrow \varphi$. If there is no inference in δ, it is an axiom $x : \varphi \Rightarrow \varphi$, and π is the corresponding axiom $\varphi \Rightarrow \varphi$ or $\bot \Rightarrow$ if $\varphi \equiv \bot$.

If δ ends in an inference rule, we distinguish cases:

1. If R is an introduction rule or $\bot_I$, the translations proceeds exactly as in the proof of **??**, and does not require Cut.

   If the last rule R of δ is an elimination rule, observe the following:

   a) A main branch of δ passes through no introduction rules, as otherwise there would be an introduction rule or $\bot_I$ followed by an elimination rule along the main branch, and δ would not be normal.

   b) Since there are no introduction rules along main branches of δ, there is only one main branch. (Only $\wedge$Intro has two major premises, so multiple main branches would have to go through the premises of a $\wedge$Intro rule.)

   c) If the main branch includes the major premise of $\vee$Elim or $\exists$Elim, there is only one such rule and it is the last one, i.e., R. (Otherwise the main branch would contain an $\vee$Elim or $\exists$Elim rule where the conclusion of the rule is major premise of an elimination rule, and the proof would not be normal as we could apply a permutation conversion.)

   d) No elimination rules other than $\vee$Elim and $\exists$Elim discharge assumptions, i.e., change the left context of sequents. $\vee$Elim and $\exists$Elim only discharge assumptions above a minor premise, i.e., change the left context only of their minor premises. In other words, if a sequent S_i on the main branch of δ has context Γ_1, the conclusion S_{i+1} of the inference of which S_i is the major premise has context $\Gamma_1' \supseteq \Gamma_1$.

   e) Consequently, if $x : \chi \Rightarrow \chi$ is the topmost sequent S_1 in the main branch, $x : \chi \in \Gamma$.

   We now distiguish cases according to the form of χ. Since every S_i in the main branch is major premise of an elimination rule, this also applies to $x : \chi \Rightarrow \chi$.

2. $\chi \equiv \theta \wedge \alpha$. Then δ has the form

$$\dfrac{\dfrac{x : \theta \wedge \alpha \Rightarrow \theta \wedge \alpha}{x : \theta \wedge \alpha \Rightarrow \theta}\,\wedge\mathrm{Elim}_1}{\begin{array}{c}\vdots\\ x : \theta \wedge \alpha, \Gamma_1 \Rightarrow \varphi\end{array}}$$

   The main branch includes $x : \theta \wedge \alpha \Rightarrow \theta$ contains no major premise of $\rightarrow$Intro, $\neg$Intro nor a minor premise of $\vee$Elim or $\exists$Elim, the context formulas in $x : \theta \wedge \alpha$ remain unchanged along the main branch. This

explains why the context of the end-sequent must contain $x : \theta \wedge \alpha$. Moreover, we can turn the derivation δ into a derivation δ_1 by replacing the sub-derivation ending in $\wedge$Elim by $x : \theta \Rightarrow \theta$ and replacing $x : \theta \wedge \alpha$ in the context of every sequent along the main branch by $x : \theta$, i.e., turn

$$\dfrac{\dfrac{x : \theta \wedge \alpha \Rightarrow \theta \wedge \alpha}{x : \theta \wedge \alpha \Rightarrow \theta}\,\wedge\mathrm{Elim}_1}{\vdots}\qquad\qquad \dfrac{x : \theta \Rightarrow \theta}{\vdots}$$
$$x : \theta \wedge \alpha, \Gamma_1 \Rightarrow \varphi \qquad \text{into} \qquad x : \theta, \Gamma_1 \Rightarrow \varphi$$

The induction hypothesis applies to δ_1, since the number of inferences in δ is the number of inferences in δ' plus 1.

By induction hypothesis, there is a **G2i**-derivation π_1 of $x : \theta, \Gamma_1 \Rightarrow \varphi$. We obtain the derivation π by applying $\wedge$L:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \theta, \Gamma_1' \Rightarrow \varphi\end{array}}{\theta \wedge \alpha, \Gamma_1' \Rightarrow \varphi}\,\wedge\mathrm{L}$$

If $\varphi \equiv \bot$ we insert a WR, e.g,

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \theta, \Gamma_1' \Rightarrow\end{array}}{\theta \wedge \alpha, \Gamma_1' \Rightarrow}\,\wedge\mathrm{L}}{\theta \wedge \alpha, \Gamma_1' \Rightarrow \varphi}\,\mathrm{WR}$$

3. If $\chi \equiv \theta \vee \alpha$, it must be the major premise of $\vee$Elim, and this inference must be the last inference R on the main branch. In other words, δ has the form:

$$\dfrac{x : \theta \vee \alpha \Rightarrow \theta \vee \alpha \qquad \begin{array}{c}\vdots\,\delta_1\\ y : \theta, \Gamma_1 \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\,\delta_2\\ z : \alpha, \Gamma_2 \Rightarrow \varphi\end{array}}{x : \theta \vee \alpha, \Gamma_1, \Gamma_2 \Rightarrow \varphi}\,\vee\mathrm{Elim}$$

The induction hypothesis applies to the derivations δ_1 and δ_2; we obtain **G2i**-derivations π_1 and π_2 of $\theta, \Gamma_1' \Rightarrow \varphi$ and $\alpha, \Gamma_2' \Rightarrow \varphi$. We apply $\vee$L to get π:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \theta, \Gamma_1' \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \alpha, \Gamma_2' \Rightarrow \varphi\end{array}}{\theta \vee \alpha, \Gamma_1', \Gamma_2' \Rightarrow \varphi}\,\vee\mathrm{L}$$

Should $\vee$Elim not discharge $y : \theta$ or $z : \alpha$ (i.e., these aren't present in the context of the minor premises), or should φ happen to be $\bot$, we have to add some WL and WR. For instance,

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{\Gamma_1' \Rightarrow}\,\mathrm{WL}}{\theta, \Gamma_1' \Rightarrow} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{\Gamma_2' \Rightarrow}\,\mathrm{WL}}{\alpha, \Gamma_2' \Rightarrow}}{\theta \vee \alpha, \Gamma_1', \Gamma_2' \Rightarrow}\,\vee\mathrm{L}}{\theta \vee \alpha, \Gamma_1', \Gamma_2' \Rightarrow \varphi}\,\mathrm{WR}$$

4. $\chi \equiv \theta \to \alpha$. Then δ has the form

$$\dfrac{\dfrac{x : \theta \to \alpha \Rightarrow \theta \to \alpha \qquad \dfrac{\vdots\,\delta_1}{\Gamma_1 \Rightarrow \theta}}{x : \theta \to \alpha, \Gamma_1 \Rightarrow \alpha}\,{\to}\mathrm{Elim}}{\begin{array}{c}\vdots\,\delta_2\\ x : \theta \to \alpha, \Gamma_1, \Gamma_2 \Rightarrow \varphi\end{array}}$$

Here note that since the main branch, which includes $x : \theta \to \alpha, \Gamma_1 \Rightarrow \alpha$, contains no major premise of $\to$Intro, $\neg$Intro nor a minor premise of $\vee$Elim or $\exists$Elim, the context formulas in $x : \theta \to \alpha, \Gamma_1$ remain unchanged along the main branch. This explains why the context of the end-sequent must contain $x : \theta \to \alpha, \Gamma_1$. Moreover, we can turn the derivation fragment δ_2 into a correct derivation δ_2' by replacing the sub-derivation ending in $\to$Elim by $x : \alpha \Rightarrow \alpha$ and replacing $x : \theta \to \alpha, \Gamma_1$ in the context of every sequent along the main branch by $x : \alpha$, i.e., turn

$$\begin{array}{c} x : \theta \to \alpha, \Gamma_1 \Rightarrow \alpha \\ \vdots\,\delta_2 \\ x : \theta \to \alpha, \Gamma_1, \Gamma_2 \Rightarrow \varphi \end{array} \quad \text{into} \quad \begin{array}{c} x : \alpha \Rightarrow \alpha \\ \vdots\,\delta_2' \\ x : \alpha, \Gamma_2 \Rightarrow \varphi \end{array}$$

The induction hypothesis applies to δ_1 and δ_2', since the number of inferences in δ is the sum of the numbers of inferences in δ_1 and δ_2, plus 1 to account for the $\to$Elim inference at the start of the main branch. (If that inference is also the last inference R in δ then δ_1 contains 0 inferences.)

By induction hypothesis, there are **G2i**-derivations π_1 of $\Gamma_1 \Rightarrow \theta$ and π_2 of $\alpha, \Gamma_1' \Rightarrow \varphi$. We obtain the derivation π by applying $\to$L:

$$\dfrac{\dfrac{\vdots\,\pi_1}{\Gamma_1' \Rightarrow \theta} \qquad \dfrac{\vdots\,\pi_2}{\alpha, \Gamma_2' \Rightarrow \varphi}}{\theta \to \alpha, \Gamma_1', \Gamma_2' \Rightarrow \varphi}\,{\to}\mathrm{L}$$

If θ and/or $\varphi \equiv \bot$ we insert one or two WR, e.g,

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{\Gamma_1' \Rightarrow}}{\Gamma_1' \Rightarrow \theta}\,\mathrm{WR} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{\alpha, \Gamma_2' \Rightarrow}}{\alpha, \Gamma_2' \Rightarrow \varphi}\,\mathrm{WR}}{\theta \to \alpha, \Gamma_1', \Gamma_2' \Rightarrow \varphi}\,{\to}\mathrm{L}$$

The remaining cases are treated similarly and are left as exercises. □

Corollary nor.3. *Every formula occuring in a normal* **N1i** *or* **N2i**-*derivation is a sub-formula of the end-formula or end-sequent.*

Proof. By Proposition nor.2, every normal **N2i**-derivation can be translated into a **G2i**-derivation. An inspection of the proof shows that every formula occurring in the **N2i**-derivation also occurs in the **G2i**-derivation. **G2i**, lacking the Cut rule, has the sub-formula property. □

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Bibliography